Stability of hybrid linear stochastic systems

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Outline

- Slowly varying linear stochastic systems
- Jump-linear stochastic systems
- Linear stochastic systems with random state resetting
Recursive ML

Continuous-time RML methods:


The BMP theory

A. Benveniste, M. Métivier, and P. Priouret, Adaptive algorithms and stochastic approximations

Developed for discrete time SA in a Markovian setting.

Objectives: analyze RML via the extension of the BMP theory:

We will focus on a single aspect of BMP theory: stability of TV hybrid linear stochastic systems.
The basic linear model

Consider a linear stochastic state-space system

\[ dX_t = A(\theta)X_t dt + B(\theta)dw_t, \quad (1) \]

with \(-\infty < t < +\infty\), where

- \( X_t \in \mathbb{R}^k \) is the state-vector,
- \( w_t \in \mathbb{R}^l \) is a standard Wiener-process over \((\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})\),
- \( \theta \in D \subset \mathbb{R}^p \), where \( D \) is an open set.

**Condition 1** The matrix \( A(\theta) \) is stable for all \( \theta \in D \), and both \( A(\theta), B(\theta) \) are continuous in \( \theta \).
For any fixed $X_t = X$ and $\theta$ we can have a $\mathbb{R}^p$-valued instantaneous observation

$$H(X, \theta)dt + G(X, \theta)dw_t,$$

where $H, G$ measurable functions.

Let the unique stationary distribution be

$$\mu_\theta,$$

and let the unique causal stationary solution be

$$X_t^*(\theta).$$
The basic estimation problem

Following BMP: find $\theta = \theta^* \in D$ such that

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T (H(X^*_t(\theta), \theta)dt + G(X^*_t(\theta), \theta)dw_t) = 0. \quad (2)$$

Equivalently: solve the algebraic equation

$$h(\theta) = \int_{\mathbb{R}^k} H(x, \theta)\mu_\theta(dx) = 0. \quad (3)$$
A recursive procedure

Assume that the Jacobian of \( h \) at \( \theta = \theta^* \) is stable. Define:

\[
d\theta_t = \frac{1}{t + T_0} (H(X_t, \theta_t)dt + G(X_t, \theta_t)dw_t)
\]

\[
dX_t = A(\theta_t)X_t dt + B(\theta_t)dw_t.
\]

Here \( X_t \) is on-line approximation of \( X^*_t(\theta_t) \).
The stopped process

The above procedure would be stopped, say at $\tau$, if:

- $\theta_t$ hits the boundary of a truncation domain $Q \subset D$
- $d\theta_t$ exceeds a fixed threshold:

$$\frac{1}{t + T_0} \left( |H(X_t, \theta_t)| + \|G(X_t, \theta_t)\| \right) \geq \delta > 0.$$ 

Thus $\theta_t$ is slowly varying in $[0, \tau]$ in a stochastic sense.
Condition A5 of BMP would translate to: for any $q \geq 1$

$$\sup_{t \geq 0} \mathbb{E} \left[ |X_t|^q \chi_{(t \leq \tau)} \right] \leq K$$

(4)

for some finite $K$, if the rate of change $\delta$ is sufficiently small.

This amounts to a stochastic extension of Desoer’s thm:
Resetting

Following Delyon’s LN, 2000, LG + Matyas, CDC’07: let $\tau_1 = \tau$, and redefine

$$\theta_\tau = \theta_0.$$ 

Restart the process with the current time. In general set:

$$\tau_{k+1} = \inf \left\{ t > \tau_k : \theta_t \notin Q, \text{ or } \frac{1}{t + T_0} \left( |H(X_t, \theta_t)| + \|G(X_t, \theta_t)\| \right) > \delta \| \right\}.$$ 

Question: can we paste together the inequalities given by the stochastic Desoer’s theorem in $[\tau_k, \tau_{k+1}]$?
Time-varying system

Consider a linear time-varying (TV) dynamics

\[ dX_t = A(\theta_t)X_t \, dt + B(\theta_t) \, dw_t, \]  \hspace{1cm} (5)

defined over \([0, \tau]\), where \(\tau > 0\) is a stopping time. Here \(\theta_t \in Q \subset D\), with \(Q\) compact, it is an Ito-process, say

\[ d\theta_t = \beta_t \, dt + \sigma_t \, dw_t. \]  \hspace{1cm} (6)

The systems is slowly varying in a stochastic sense:

\[ |b_t| + \|\sigma_t\| \leq \delta \]  \hspace{1cm} (7)

with some \(\delta > 0\) for all \(t \in [0, \tau]\).
Theorem 1  Assume Condition 1 to hold, and let \( \theta_t \) be slowly varying in the stochastic sense. Let \( q \geq 0 \), and \( \mathbb{E} \left[ |X_0|^q \right] < \infty \). If \( \delta \) is small enough, then

\[
\sup_{t \geq 0} \mathbb{E} \left[ |X_t|^q \chi_{(t \leq \tau)} \right] < \infty.
\]
Random horizons

Let $0 < \tau_i$ be an increasing sequence of stopping times, with

$$\tau_i < \tau_{i+1} \quad \text{if} \quad \tau_i < \infty.$$  

Let $N_t$ be the counting process associated with $\tau_i$, s.t. $N_t < \infty$ a.s. for any $t > 0$.

Task: paste together the inequalities of stochastic Desoer’s theorem formulated on $[\tau_i, \tau_{i+1})$, i.e.

$$\sup_{\tau_i+t \geq 0} \mathbb{E} \left[ |X_{\tau_i+t}|^q \chi_{(\tau_i+t \leq \tau_{i+1})} \right] < \infty.$$  

(9)
Let $\theta_t \in Q$ be an adapted, piecewise continuous cadlag process with jumps at $\tau_i$ so that

$$\theta_{\tau_i} = \theta_0 \in \text{int } Q.$$ 

**Condition 2** $\theta_t$ is an Ito-process in $[\tau_i, \tau_{i+1})$, thus

$$d\theta_t = \beta_t dt + \sigma_t dw_t + (\theta_0 - \theta_t-) dN_t. \quad (10)$$

Moreover, with some $\delta > 0$

$$\sup_t (|b_t| + \|\sigma_t\|) \leq \delta.$$
Theorem 2 Assume that Conditions 1 and 2 hold. Let \( q \geq 0 \), and assume that \( E |X_0|^q < \infty \). Then there exist a \( \delta_0 > 0 \) such that if \( \delta \leq \delta_0 \) then

\[
\sup_{t \geq 0} E |X_t|^q < \infty.
\]

The proof is based on exploiting the asymmetry of jumps.
A key lemma

Lemma 1 Assume Condition 1 to hold. Then there is a smooth, matrix-valued function $P = P(\theta), \theta \in D$, with $P = P^T, P \geq 0$, solving

$$P(\theta) A(\theta) + A^T(\theta) P(\theta) \leq -\alpha P(\theta)$$

(11)

with some $\alpha > 0$, for all $\theta \in Q$, such that

$$P(\theta) \geq P(\theta_0) \quad \text{for all} \quad \theta \in Q.$$
Resetting the state vector

Consider the hybrid LSS with $A$ stable,

$$dX_t = AX_t dt + Bd\omega_t + (X_0 - X_{t-}) dN_t.$$  \hspace{1cm} (12)

Question: may resetting destroy stability?

**Theorem 3** Let $N_t$ be as above. Let $q \geq 0$, and let $E |X_0|^q < \infty$. Then

$$\sup_t E \left[ |X_t|^q \right] < \infty.$$
A useful concept to organize computations:

**Definition 1** Let $V_t > 0$ be an Ito-process. $V_t$ satisfies the geometric drift condition with $\gamma$-bounded volatility, $\gamma > 0$, if

$$dV_t = V_t (\tilde{\nu}_t dt + dM_t),$$  \hspace{1cm} (13)

where $M_t$ is a martingale, such that for some $\alpha, K, L > 0$

$$\tilde{\nu}_t \leq -\alpha + L\chi_{(V_t<K)},$$ \hspace{1cm} (14)

and

$$\frac{d\langle M \rangle_t}{dt} \leq \gamma.$$
Ultimately bounded volatility

**Definition 2** $V_t$ satisfies the geometric drift condition with *ultimately* $\gamma$-bounded volatility, if the drift condition above is satisfied, and for some $K', L' > 0$

$$\frac{d \langle M \rangle_t}{dt} \leq \gamma + L' \chi(V_t < K').$$  (15)
Lemma 2  Let $V_t$ satisfy the geometric drift condition with ultimately $\gamma$-bounded bounded volatility. Then

$$\sup_{t \geq 0} \mathbb{E} [V_t] < \infty.$$ 

Lemma 3  Let $V_t \geq 1$ be as above. Then for any $q > 0$ the process $V_t^q$ also satisfies the geometric drift condition with ultimately $q^2 \gamma$-bounded volatility, if $\gamma$ is small.
A stochastic Lyapunov function

To prove the stochastic Desoer’s theorem: let $P(\theta)$ be as in Lemma 1, and set

$$V_t = (1 + X_t^T P(\theta_t) X_t) = 1 + \text{Tr} (P_t Z_t),$$

where $P_t = P(\theta_t)$ and $Z_t = X_t X_t^T$.

**Lemma 4** For any $\gamma > 0$ the process $V_t$ satisfies the geometric drift condition with ultimately $\gamma$-bounded volatility, whenever $\delta$ is sufficiently small.
Jumps in the dynamics

The problem: find an upper bound for the cumulative effects of jumps. Let $V_t$ be as above, then

$$dV_t = V_t(v_t dt + dM_t) + u_t dt + \xi_t dN_t,$$

$$v_t \leq -\alpha \quad \text{with} \quad \alpha > 0,$$

where $M$ is a martingale with bounded volatility,

$$u_t \leq c \quad \text{and} \quad \xi_t \leq 0.$$

The condition $\xi_t \leq 0$ follows from the minimality of $P(\theta_0)$. Thus the cumulative effect of jumps is negative!
State resetting

Consider a system with the dynamics state resetting:

\[ dX_t = AX_t dt + Bdw_t + (X_0 - X_{t-})dN_t, \]

where \( N_t \) is a counting process as above. Let

\[ V_t = (1 + X_t^T PX_t). \]

Then:

\[ dV_t = V_t(v_t dt + dM_t) + (V_0 - V_{t-})dN_t. \]

Here \( V_t \) would satisfy the geometric drift-condition with \( \gamma \)-bounded volatility, except for the resetting.
To have an additive dynamics consider:

\[
d \log V_t = v_t dt + dM_t - \frac{1}{2} d\langle M \rangle_t + (\log(V_0) - \log(V_{t-})) dN_t.
\]

Define the auxiliary process

\[
U_t = -\alpha' t + M_t - M_0 - \frac{1}{2} \langle M \rangle_t.
\]

Let \( T > 0 \) and let \( \tau_n \) be the last resetting prior to \( T \). Then

\[
d \log V_t \leq dU_t \quad \text{for} \quad \tau_n < t < T.
\]
Integration leads to

\[ \log V_T \leq \log K + (U_T - \inf_{t \leq T} U_t) = \log K + Y_T, \]

where

\[ Y_T = U_T - \inf_{t \leq T} U_t \]

is the so-called Skorohod reflection of \( U_t \).
If $M_t = w_t$ then

$$Y_t = \sup_{s \leq t} \left[ - (\alpha' + \frac{1}{2}) (t - s) + (w_t - w_s) \right].$$  \hspace{1cm} (17)

Now $(w_t - w_s)$ is a Wiener-process in $s$. Thus $Y_t \leq^s Y^*$ with

$$Y^* = \sup_{0 \leq u < \infty} \left[ - (\alpha' + \frac{1}{2}) u + w_u \right].$$  \hspace{1cm} (18)

$Y^*$ has exponential distribution with parameter $2\alpha' + 1$. Thus

$$\mathbb{E} \left[ e^{qY^*} \right] < \infty \quad \text{for} \quad q < 2\alpha' + 1.$$
The general case

To estimate $Y_t$: find a semi-martingale $Y'_t$ such that

$$Y_t = |Y'_t|.$$  \hfill (19)

**Theorem 4** Let $U_t$ be a cont. semi-martingale, $U_0 = 0$, let

$$Y_t = U_t - \inf_{s \leq t} (U_s).$$

Then on a suitable enlargement of $(\Omega, \mathcal{F}, P)$ there is a continuous semi-martingale $Y'_t$ such that (19) holds, and

$$dY'_t = \text{sign}(Y'_t) \ dU_t.$$
THANK YOU FOR YOUR ATTENTION