On the McMillan degree of general Linear Fractional Transformations

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1 Abstract

We investigate here the interpolation conditions connected to an interpolating function \( Q \) obtained as a Linear Fractional Transformation of another function \( S \). In general, the degree of \( Q \) is equal to the number of interpolating condition plus the degree of \( S \). We show that, if the degree of \( Q \) is strictly less than this quantity, there is number of complementary interpolating condition which has to be satisfied by \( S \). This induces a partitioning of the interpolating conditions in two sets. We consider here the case where these two sets are disjoint. The reasoning can also be reversed (i.e. from \( S \) to \( Q \)).

2 Introduction

The linear fractional transformation – denoted by \( Q = T_\Theta(S) \) (or defined explicitly as: \( Q = (S\Theta_{12} + \Theta_{22})(S\Theta_{11} + \Theta_{21})^{-1} \), where \( \Theta = \begin{pmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{pmatrix} \)) of – possibly – matrix valued functions plays an important role in various problems of system theory, like the Youla-Kucera parametrization of stabilizing controllers of a given plant (see [4]) or the general solution of the Nevanlinna-Pick interpolation problem (see [1, 3]), to mention only two. In both cases the solutions are given as results of a linear fractional transformation applied to an arbitrary function in a given set.

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In the above transformation, it generically happens that \( \deg Q = \deg S + \deg \Theta \). The special case \( \deg Q = \deg \Theta \) was considered by Kimura (see [7]) and Georgiou ([5]) and by Byrnes, Georgiou and Lindquist with the added positive realness constraint on the interpolating function (see [2]). In the present paper we analyze the interpolation properties of the functions \( Q \) and \( S \), connected by a linear fractional transformation determined by a given matrix-valued rational function \( \Theta \), when a drop in the degree of \( Q \) occurs. We show that, in this case, two sets of interpolation problems can be connected to a given function \( \Theta \), where the interpolation nodes are given by the poles of \( \Theta \) and its inverse and the functions \( Q, S \) "together" always "essentially" provide solutions of these interpolation problems. Here the together means that the poles of \( \Theta \) (and of \( \Theta^{-1} \)) should be split into two parts and on one of these parts the function \( Q \) gives a solution of the corresponding interpolation problem, on the other part the function \( S \). Note that this splitting may depend on the functions \( Q \) and \( S \). Especially, if we choose a function \( S \) not being a solution of the interpolation problem determined by \( \Theta \) on any subsets of its poles then the resulting function \( Q \) should satisfy the interpolation conditions of the whole set of poles of \( \Theta \). This property is used in the construction of all solutions of the Nevanlinna-Pick interpolation problem given by a linear fractional transformation.

The "essentially" above means that extra care should be taken into account when some of the poles of the functions \( \Theta, Q, S \) coincide. For sake of simplicity this general case will be treated elsewhere.

3 General linear fractional transformation

Given the interpolation data \( A, H, L \), it is well known (see [1]) that all functions \( Q \) which interpolate these conditions, i.e. satisfy the relation

\[
(Q(s)H + L)(sI + A^*)^{-1} \text{ is analytic in } \sigma(-A^*)
\]

(3.1)
can be represented by means of a Linear Fractional Transformation \( \Theta = \begin{pmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{pmatrix} \) with the following minimal realization:

\[
\Theta = \begin{pmatrix} A & U & V \\ H & I & 0 \\ L & 0 & I \end{pmatrix}
\]

(3.2)

where we assume that the rational functions \( Q \) and \( S \) are related by

\[
Q = T_{\Theta}(S) = (S\Theta_{12} + \Theta_{22})^{-1}(S\Theta_{11} + \Theta_{21}) .
\]

(3.3)
The relation can be written as:

\[
(S\Theta_{11} + \Theta_{21}) = (S\Theta_{12} + \Theta_{22})Q
\]

or

\[
[(S\Theta_{11} + \Theta_{21}), (S\Theta_{12} + \Theta_{22})] \begin{bmatrix} I \\ -Q \end{bmatrix} = 0
\]
and thus

\[ [S, I] \begin{pmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{pmatrix} \begin{bmatrix} I \\ -Q \end{bmatrix} = 0 \]  

(3.4)

Denoting the minimal realizations of \( Q \) and \( S \) as

\[ Q = \begin{bmatrix} A_Q & B_Q \\ C_Q & D_Q \end{bmatrix}, \quad S = \begin{bmatrix} A_S & B_S \\ C_S & D_S \end{bmatrix} \]

the realization expression of (3.4) can easily seen to be:

\[
\begin{bmatrix}
A_Q & A & B_Q \\
-V C_Q & B_H & U - V D_Q \\
-C_Q & D_S H + L & C_S & D_S - D_Q
\end{bmatrix} = 0
\]  

(3.5)

giving immediately that \( D_S = D_Q \). Therefore we shall use the notation \( D \) for both.

Let denote the McMillan-degrees of \( \Theta \), \( Q \) and \( S \) by \( n_\Theta \), \( n_Q \) and \( n_S \), respectively.

It is well known that, in general, if we start with an arbitrary \( S \), we have \( n_Q = n_S + n \). The aim of this paper is to characterize those situation where \( n_Q < n_S + n \). The following two lemmata express the connections between \( S \) and \( Q \) provided by the linear fractional transformation (3.3) in terms of their realizations. Their proofs is straightforward and it is omitted.

**Lemma 1.** Let \( \Theta \) be a LFT, and let

\[
S = \begin{bmatrix} A_S & B_S \\ C_S & D \end{bmatrix}.
\]  

(3.6)

Then \( Q = T_\Theta(S) \) has the following (possibly non-minimal) realization:

\[
Q = \begin{bmatrix} A - V(DH + L) & -VC_S & U - VD \\ B_S H & A_S & B_S \\ DH + L & C_S & D \end{bmatrix}
\]  

(3.7)

(Let us observe that this is immediate from (3.5) using the output-injection determined by \[ \begin{bmatrix} 0 \\ -V \\ 0 \end{bmatrix} \].)

**Lemma 2.** Let \( \Theta \) be a general LFT and let

\[
Q = \begin{bmatrix} A_Q & B_Q \\ C_Q & D \end{bmatrix}.
\]  

(3.8)
Then $S = T_{\bar{\sigma}}^{-1}(Q)$ has the following (possibly non-minimal) realization:

$$
S = \begin{pmatrix}
A_Q & B_QH \\
VC_Q & A - (U - VD)H \\
C_Q & -(U - VD)H \\
DH + L & D
\end{pmatrix}
$$

(3.9)

(This is again immediate from (3.5) applying the feedback determined by $[0, -H, 0]$.)

The following result will be needed.

**Theorem 3.** Let $F(s) = D + C(sI - A)^{-1}B$ be a rational function.

(i) Assume that there exists a - possibly matrix-valued - function

$$
\phi(s) = C\phi(sI - A\phi)^{-1}B\phi + \psi(s),
$$

where $\psi$ is a matrix-valued polynomial and $(A\phi, B\phi)$ is a controllable pair such that $F\phi$ is analytic at the eigenvalues of $A\phi$ and assume, moreover, that the pair $(C, A)$ is observable.

Then there exists a matrix $\Pi$ such that

$$
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix}
\begin{bmatrix}
\Pi \\
\Pi A\phi
\end{bmatrix} = \begin{bmatrix}
\Pi A\phi \\
0
\end{bmatrix}.
$$

(3.10)

(ii) Assume that the matrices $A\phi, C\phi, \Pi$ satisfy the equation (3.10), where $(C\phi, A\phi)$ is an observable pair, $(A, B)$ is a controllable pair. Then there exists a matrix polynomial $\psi$ such that for

$$
\phi(s) = C\phi(sI - A\phi)^{-1} + \psi(s)
$$

the function $F\phi$ is analytic at the eigenvalues of $A\phi$. (Or more specially, $\psi$ can be chosen in such a way that $F\psi$ be a polynomial.)

If $A$ and $A\phi$ have no common eigenvalues then equation (3.10) implies that

$$
F(s)C\phi(sI - A\phi)^{-1} = C(sI - A)^{-1}\Pi,
$$

which is analytic at the eigenvalues of $A\phi$.

For the proof we refer to [6].

### 3.1 Controllability and observability analysis

We investigate now the controllable and unobservable subspaces of the realizations (3.7) and (3.9) of $Q$ and $S$, respectively.
We start with the unobservable subspace of realization (3.7) of $Q$. Denote by $O_Q$ this unobservable subspace. Set $d_{o,Q} = \dim (O_Q)$.

By output injection we can make the dynamics in the realization lower triangular. Thus if $\begin{bmatrix} \alpha_Q \\ \beta_Q \end{bmatrix}$ is a suitably partitioned matrix, then using the PBH-test, its columns lie in the unobservable subspace of realization (3.7) if and only if

$$[DH + L, C_S] \begin{bmatrix} \alpha_Q \\ \beta_Q \end{bmatrix} = 0 \quad (3.12)$$

$$\begin{bmatrix} A \\ B_S H \\ A_S \end{bmatrix} \begin{bmatrix} \alpha_Q \\ \beta_Q \end{bmatrix} = \begin{bmatrix} \alpha_Q \\ \beta_Q \end{bmatrix} \Gamma_Q \quad (3.13)$$

for some matrix $\Gamma_Q$, which means that

$$A \alpha_Q = \alpha_Q \Gamma_Q \quad (3.14)$$

and

$$\begin{bmatrix} A_S \\ B_S H \\ C_S \end{bmatrix} \begin{bmatrix} \beta_Q \\ \alpha_Q \end{bmatrix} = \begin{bmatrix} \beta_Q \Gamma_Q \\ 0 \end{bmatrix} \quad (3.15)$$

Let us assume that the columns of $\begin{bmatrix} \alpha_Q \\ \beta_Q \end{bmatrix}$ form a basis in $O_Q$. This matrix is obviously not uniquely defined. Multiplying it from the right with a nonsingular matrix (and modifying $\Gamma_Q$ correspondingly) we get another (maximal) solution of equations (3.12), (3.13). Note that the observability of the pair $(C_S, A_S)$ implies that the columns of $\alpha_Q$ are linearly independent. Invoking identity (3.14) we obtain from this that the spectrum of $\Gamma_Q$ is a subset of that of $A$. I.e.

$$\sigma (\Gamma_Q) \subseteq \sigma (A) \quad (3.16)$$

Similarly, for the uncontrollability analysis of the realization (3.7) of $Q$. Denote by $C_Q$ the controllable subspace of (3.7). Set $d_{c,Q} = \dim (C_Q)$. We can make the dynamics matrix upper triangular by feedback; thus the rows of a suitably partitioned matrix $\begin{bmatrix} \xi_Q, \eta_Q \end{bmatrix}$ are in the orthogonal complement of the controllable subspace of (3.7) if and only if

$$[\xi_Q, \eta_Q] \begin{bmatrix} U - V D \\ B_S \end{bmatrix} = 0 \quad (3.17)$$

$$[\xi_Q, \eta_Q] \begin{bmatrix} A - U H - V L \\ 0 \\ VC_S \\ A_S \end{bmatrix} = \Lambda_Q [\xi_Q, \eta_Q] \quad (3.18)$$

which means that

$$\xi_Q (A - U H - V L) = \Lambda_Q \xi_Q \quad (3.19)$$

and

$$[\eta_Q, \xi_Q] \begin{bmatrix} A_S \\ -VC_S \\ B_S \\ U - V D \end{bmatrix} = [\Lambda_Q \eta_Q, 0] \quad (3.20)$$

Let us assume that the rows of $[\xi_Q, \eta_Q]$ form a basis in the orthogonal complement of $C_Q$. As before, in this case the controllability of the pair $(A_S, B_S)$ implies that
the rows of $\xi_Q$ are linearly independent. Invoking identity (3.19) we obtain from this that the spectrum of $\Lambda Q$ is a subset of that of $A - UH - VL$. I.e.

$$\sigma(\Lambda Q) \subset \sigma(A - UH - VL).$$

(3.21)

A completely similar analysis holds for the unobservable and controllable subspaces of realization (3.9) of $S$. Thus if $\begin{bmatrix} \beta_S \\ \alpha_S \end{bmatrix}$ is a suitably partitioned matrix, its columns lie in the unobservable subspace – denoted by $O_S$, $d_{o,S} = \dim(O_S)$ – of realization (3.9) if and only if

$$[C_Q, DH + L] \begin{bmatrix} \beta_S \\ \alpha_S \end{bmatrix} = 0$$

(3.22)

and

$$\begin{bmatrix} A_Q & B_QH \\ VC_Q & A - (U - VD)H \end{bmatrix} \begin{bmatrix} \beta_S \\ \alpha_S \end{bmatrix} = \begin{bmatrix} \beta_S \\ \alpha_S \end{bmatrix} \Gamma_S$$

(3.23)

for some matrix $\Gamma_S$.

Let us assume that the columns of $\begin{bmatrix} \beta_S \\ \alpha_S \end{bmatrix}$ form a basis in the $O_S$.

Since we can again lower triangularize the dynamics in the realization by output injection, we obtain that

$$(A - UH - VL)\alpha_S = \alpha_S \Gamma_S$$

(3.24)

and

$$\begin{bmatrix} A_Q & B_QH \\ VC_Q & DH + L \end{bmatrix} \begin{bmatrix} \beta_S \\ \alpha_S \end{bmatrix} = \begin{bmatrix} \beta_S \Gamma_S \\ 0 \end{bmatrix}.$$  

(3.25)

Again in this case the observability of the pair $(C_Q, A_Q)$ implies that the columns of $\alpha_S$ are linearly independent. Invoking identity (3.24) we obtain from this that the spectrum of $\Gamma_S$ is a subset of that of $A - UH - VL$. I.e.

$$\sigma(\Gamma_S) \subset \sigma(A - UH - VL).$$

(3.26)

Similarly, for the controllability analysis of (3.9). Denote by $C_S$ the controllable subspace of (3.9). Set $d_{c,S} = \dim(C_S)$. We can make the dynamics matrix upper triangular by feedback; thus the rows of a suitably partitioned matrix $[\eta_S, \xi_S]$ are in the orthogonal complement of the controllable subspace of (3.9) if and only if

$$[\eta_S, \xi_S] \begin{bmatrix} B_Q \\ -(U - VD) \end{bmatrix} = 0$$

(3.27)

$$[\eta_S, \xi_S] \begin{bmatrix} A_Q & 0 \\ VC_Q & A \end{bmatrix} = \Lambda_S[\eta_S, \xi_S]$$

(3.28)

which means that

$$\xi_S A = \Lambda_S \xi_S$$

(3.29)

and

$$[\eta_S, \xi_S] \begin{bmatrix} A_Q & B_Q \\ VC_Q & -(U - VD) \end{bmatrix} = [\Lambda_S \eta_S, 0]$$

(3.30)
Let us assume that the rows of \([\eta_S, \xi_S]\) form a basis in the orthogonal complement of \(C_S\). In this case the controllability of the pair \((A_Q, B_Q)\) implies that the rows of \(\xi_S\) are linearly independent. Invoking identity (3.29) we obtain from this that the spectrum of \(\Lambda_S\) is a subset of that of \(A\). I.e.

\[
\sigma(\Lambda_S) \subset^* \sigma(A).
\]  

**Lemma 4.** Suppose that

\[
\sigma(A) \cap \sigma(A - UH - VL) = \emptyset.
\]

Then

(i) \(O_Q \subset C_Q\),

(ii) \(O_S \subset C_S\),

**Proof.** Note that the statements (i) and (ii) can be expressed as:

\[
\begin{align*}
[\xi_Q, \eta_Q] \begin{bmatrix} \alpha_Q \\ \beta_Q \end{bmatrix} &= 0, \\
[\eta_S, \xi_S] \begin{bmatrix} \beta_S \\ \alpha_S \end{bmatrix} &= 0.
\end{align*}
\]  

(3.32)

Now multiplying the dynamics of (3.9) by \([\eta_S, \xi_S]\) and \(\begin{bmatrix} \beta_S \\ \alpha_S \end{bmatrix}\) on the left and, respectively on the right hand side, we obtain, in view of (3.23) and (3.28),

\[
\begin{align*}
[\eta_S, \xi_S] \begin{bmatrix} A_Q \\ VC_Q \end{bmatrix} A - (U - VD)H \begin{bmatrix} \beta_S \\ \alpha_S \end{bmatrix} &= \Lambda_S[\eta_S, \xi_S] \begin{bmatrix} \beta_S \\ \alpha_S \end{bmatrix} = [\eta_S, \xi_S] \begin{bmatrix} \beta_S \\ \alpha_S \end{bmatrix} \Gamma_S.
\end{align*}
\]

Now, by construction, using equations (3.31) and (3.26) \(\sigma(\Lambda_S) \subset^* \sigma(A)\) and \(\sigma(\Gamma_S) \subset^* \sigma(A - UH - VL)\) and thus their spectra are disjoint, yielding the conclusion for (ii). Similarly for (i).

Using the possibly nonminimal realizations (3.7) and (3.9) of \(Q\) and \(S\), respectively, we get the following immediate corollary.

**Corollary 5.** Suppose that

\[
\sigma(A) \cap \sigma(A - UH - VL) = \emptyset.
\]

Then

(i) \(n_Q = d_{c,Q} - d_{o,Q}\),

(ii) \(n_S = d_{c,S} - d_{o,S}\).

Note that equation (3.14) means that \(\text{Im} (\alpha_Q)\) is \(A\)-invariant and on it the action of the matrix \(A\) is given by \(\Gamma_Q\). In other words in an appropriate basis the
form of $A$ is lower block-triangular where one of the diagonal entries is given by $\Gamma_Q$. Similarly, (3.29) gives that the rows space of $\eta_S$ is invariant under the right action defined by $A$ and the matrix $\Lambda_S$ determines the (right)-action of $A$ on it. In other words $A$ has – in an appropriate basis – a lower block-triangular form where one of the diagonal entries is $\Lambda_S$. The following theorem shows that this special basis can be chosen in such a way that the diagonal entries of $A$ are exactly $\Lambda_S$ and $\Gamma_Q$.

Similar statement holds for the matrix $A - UH - VL$ – with the diagonal entries $\Lambda_Q$ and $\Gamma_S$.

Moreover, we can interpret the above results as additional interpolation conditions on the functions $Q$ and $S$.

**Theorem 6.** Let $Q$ and $S$ be related as in (3.5) and $\Lambda_Q, \Lambda_S, \Gamma_Q, \Gamma_S$ and $\alpha_Q, \xi_Q, \alpha_S, \xi_S$ be defined as above. There the row vectors of $\xi_S$ can be extended to a nonsingular transformation $R = \begin{bmatrix} \xi_S \\ \zeta_Q \end{bmatrix}$ such that its inverse $R^{-1} = [\rho_S, \alpha_Q]$ gives an extension of the column vectors of $\alpha_Q$ to a basis and

$$RAR^{-1} = \begin{bmatrix} \Lambda_S & 0 \\ \zeta_Q \Lambda \rho_S & \Gamma_Q \end{bmatrix}. \tag{3.33}$$

Furthermore:

(i) The pair $(\alpha_Q, \Gamma_Q)$ is a right null pair for $SH + L$

(ii) The pair $(\Lambda_S, \xi_S)$ is a left null pair for $U - VQ$

Similarly, the row vectors of $\xi_Q$ can be extended to a nonsingular transformation $S = \begin{bmatrix} \xi_Q \\ \zeta_S \end{bmatrix}$ such that its inverse $S^{-1} = [\rho_Q, \alpha_S]$ gives an extension of the column vectors of $\alpha_S$ to a basis and

$$S[A - UH - VL]S^{-1} = \begin{bmatrix} \Lambda_Q \\ \zeta_S(A - UH - VL) \rho_Q & \Gamma_S \end{bmatrix}. \tag{3.34}$$

Furthermore:

(iii) The pair $(\Gamma_S, \alpha_S)$ is a right null pair for $QH + L$

(iv) The pair $(\Lambda_Q, \xi_Q)$ is a left null pair for $U - VS$

Finally

$$\dim (C_Q) = \dim (C_S), \quad \dim (O_Q^\perp) = \dim (O_S^\perp). \tag{3.35}$$

Let us remark that according to this theorem the number of (left- and right-) interpolation conditions formulated on the function $Q$ is $n_\Theta + n_Q - d_{c,S} + d_{o,Q} = n_\Theta$. Similarly for the function $S$ (i) and (iv) altogether means $n_\Theta$ number of interpolation conditions.
Proof. Let us start with a fixed realization of $Q$ given in (3.8) and construct the nonminimal realization (3.9) of $S$ and the matrices $\begin{bmatrix} \beta_S \\ \alpha_S \end{bmatrix}$ and $[\eta_S, \xi_S]$.

Since $\eta_S$ is full row-rank, we can extend it to a square invertible matrix $\tilde{R} = \begin{bmatrix} \xi_S \\ \zeta_Q \end{bmatrix}$ as above. (For the time being let us take an arbitrary extension.) Note that $\xi_S$ has $n_\Theta + n_Q - d_{c,S}$ number of rows.

Partition the inverse accordingly as $\tilde{R}^{-1} = [\rho_S, \sigma_Q]$. Now $\sigma_Q$ is of size $n_\Theta \times (d_{c,S} - n_Q)$. Then, in view of (3.29),

$$\tilde{R}A\tilde{R}^{-1} = \begin{bmatrix} \xi_S \\ \zeta_Q \end{bmatrix} A[\rho_S, \sigma_Q] = \begin{bmatrix} \xi_S \alpha S \rho_S & \xi_S \xi S \sigma Q \\ \zeta_Q \alpha S \rho_S & \zeta_Q \xi S \sigma Q \end{bmatrix} = \begin{bmatrix} \Lambda S \xi S \rho_S & \Lambda S \xi S \sigma Q \\ \zeta_Q \alpha S \rho_S & \zeta_Q \xi S \sigma Q \end{bmatrix}.$$

Now starting from the realization (3.9) of $S$ we construct a minimal realization. I.e. start with the column vectors of $\begin{bmatrix} \beta_S \\ \alpha_S \end{bmatrix}$ which are linearly independent and span the unobservable subspace $O_S$ of the realization (3.9), extend this to a basis $\begin{bmatrix} \beta_S \\ \beta_{m,S} \\ \alpha_S \\ \alpha_{m,S} \end{bmatrix}$ in $C_S$ and finally to a basis $\begin{bmatrix} \beta_{c,S} \\ \beta_S \\ \beta_{m,S} \\ \alpha_{c,S} \\ \alpha_S \\ \alpha_{m,S} \end{bmatrix}$ in the whole state-space of (3.9).

Let us introduce the notation $T = \begin{bmatrix} \beta_{c,S} & \beta_S & \beta_{m,S} \\ \alpha_{c,S} & \alpha_S & \alpha_{m,S} \end{bmatrix}$. (3.36)

Since, by construction, the row vectors of $[\eta_S, \xi_S]$ generate the orthogonal complement of $C_S$, the first block-column of $T$ can be chosen so that the first block-row of $T^{-1}$ is $[\eta_S, \xi_S]$. With this assumption, the remaining block-entries of $T^{-1}$ can be written as follows:

$$T^{-1} = \begin{bmatrix} \eta_{S} & \xi_{S} \\ \eta_{m,S} & \xi_{m,S} \end{bmatrix}.$$ (3.37)

Now

$$S \sim \begin{pmatrix} \eta_{m,S}, \xi_{m,S} & \Lambda S \xi S \rho S - (U - V D)H \beta_{m,S} \\ \xi_{m,S} & \alpha_{m,S} \end{pmatrix} \begin{pmatrix} \eta_{m,S}, \xi_{m,S} & \Lambda S \xi S \sigma Q - B_Q H \xi_{m,S} \\ \alpha_{m,S} & \alpha_{m,S} \end{pmatrix}$$

(3.38)

gives a minimal realization of $S$.

Set now $\hat{\alpha}_Q := \sigma_Q$, $\hat{\beta}_Q := -\xi_{m,S} \sigma_Q$ and $\hat{\Gamma}_Q := \zeta_Q \alpha S \sigma Q$. We claim now that the columns of $\begin{bmatrix} \hat{\alpha}_Q \\ \hat{\beta}_Q \end{bmatrix}$ are in the unobservable subspace of the realization (3.7) of $Q$ obtained from the minimal realization (3.38) of $S$ constructed above.
To verify this, we have to check that equations (3.14) and (3.15) hold for \( \tilde{\alpha}_Q, \tilde{\beta}_Q, \tilde{\Gamma}_Q \).

As for (3.14) we have:

\[
\tilde{\alpha}_Q \tilde{\Gamma}_Q = \sigma_Q \zeta Q A \sigma_Q = (I - \rho_s \xi_s)A \sigma_Q = A \sigma_Q - \rho_s \Lambda_s \xi_s \sigma_Q = A \tilde{\alpha}_Q .
\] (3.39)

Using the expressions (3.36) and (3.37) of \( T \) and \( T^{-1} \), we get

\[
(I - \alpha_r, s \xi_r, s)\sigma_Q = (\alpha_c, s \xi_s + \alpha_s \xi_o, s)\sigma_Q = \alpha_s \xi_o, s \sigma_Q
\] (3.40)

and

\[
-\beta_r, s \xi_r, s \sigma_Q = (\beta_c, s \xi_s + \beta_s \xi_o, s)\sigma_Q = \beta_s \xi_o, s \sigma_Q
\] (3.41)

using that \( \xi_s \sigma_Q = 0 \).

Let us check now equation (3.15). The last expression, together with (3.22) allows us to write:

\[
[C_Q, D H + L] \left[ \begin{array}{c} \beta_{m, s} \\ \alpha_{m, s} \end{array} \right] \hat{\beta}_Q + (D H + L) \tilde{\alpha}_Q \] (3.42)

\[
= -C_Q \beta_{m, s} \xi_{m, s} \sigma_Q + (D H + L)(\sigma_Q - \alpha_{m, s} \xi_{m, s} \sigma_Q)
\] (3.43)

\[
= [C_Q \beta_s + (D H + L)\alpha_s] \xi_{o, s} \sigma_Q = 0
\] (3.44)

from equation (3.25).

Finally, using (3.40) and (3.41) and (3.23)

\[
\left[ \begin{array}{c} A_Q \\ V C_Q \end{array} \right] \left[ \begin{array}{c} B_Q H \\ A - U H + V D H \end{array} \right] \left[ \begin{array}{c} \beta_{m, s} \\ \alpha_{m, s} \end{array} \right] \tilde{\beta}_Q
\]

\[
+ \left[ \begin{array}{c} B_Q \\ V D - U \end{array} \right] H \tilde{\alpha}_Q - \tilde{\beta}_Q \tilde{\Gamma}_Q
\] (3.45)

\[
= - \left[ \begin{array}{c} A_Q \\ V C_Q \end{array} \right] \left[ \begin{array}{c} B_Q H \\ A - U H + V D H \end{array} \right] \left[ \begin{array}{c} \beta_{m, s} \\ \alpha_{m, s} \end{array} \right] \xi_{m, s} \sigma_Q
\]

\[
+ \left[ \begin{array}{c} B_Q \\ V D - U \end{array} \right] H \sigma_Q + \xi_{m, s} \sigma_Q \tilde{\Gamma}_Q
\] (3.46)

\[
= - \left[ \begin{array}{c} A_Q \\ V C_Q \end{array} \right] \left[ \begin{array}{c} \beta_{m, s} \\ \alpha_{m, s} \end{array} \right] \xi_{m, s} \sigma_Q
\]

\[
\left[ \begin{array}{c} \phi_Q \\ \psi_Q \end{array} \right] + \left[ \begin{array}{c} B_Q \\ V D - U \end{array} \right] H(I - \alpha_{m, s} \xi_{m, s})\sigma_Q + \xi_{m, s} A \sigma_Q
\]

\[
= \left[ \begin{array}{c} A_Q \\ V C_Q \end{array} \right] \left[ \begin{array}{c} \beta_{s} \xi_{o, s} \\ \alpha_{s} \xi_{o, s} - I \end{array} \right] \sigma_Q
\]

\[
+ \left[ \begin{array}{c} B_Q \\ V D - U \end{array} \right] H \alpha_{s} \xi_{o, s} \sigma_Q + \xi_{m, s} A \sigma_Q
\] (3.47)

\[
= \left[ \begin{array}{c} A_Q \\ V C_Q \end{array} \right] \left[ \begin{array}{c} \beta_{s} \\ \alpha_{s} \end{array} \right] + \left[ \begin{array}{c} B_Q \\ V D - U \end{array} \right] H \alpha_{s} \xi_{o, s} \sigma_Q
\]
\[
\begin{aligned}
\eta_{m,S}, \xi_{m,S} \left[ \frac{A_Q}{V C_Q} \begin{array}{c} B_Q H \\ A - (U - V D) H \end{array} \right] \begin{bmatrix} \beta_S \\ \alpha_S \end{bmatrix} \xi_{o,s} \sigma_Q \\
= \eta_{m,S}, \xi_{m,S} \left[ \begin{array}{c} \beta_S \\ \alpha_S \end{array} \right] \Gamma_S \xi_{o,s} \sigma_Q = 0.
\end{aligned}
\]

Comparing now with (3.14) and (3.15), we obtain that the columns of \(\hat{\alpha}_Q, \hat{\beta}_Q\) are in \(O_Q\). Especially, using that the columns of \(\hat{\alpha}_Q = \sigma_Q\) are linearly independent, \(d_{c,S} - n_Q \leq d_{o,Q}\). (3.48)

Similar argument gives that \(d_{c,Q} - n_S \leq d_{o,S}\). (3.49)

Taking the sum and rearranging \(d_{c,S} + d_{c,Q} - d_{o,Q} - d_{o,S} \leq n_Q + n_S\).

But according the Corollary 5 \(n_Q + n_S = d_{c,Q} - d_{o,Q} + d_{c,S} - d_{o,S}\), i.e. equality should hold. Consequently in (3.48) and (3.49) are equalities, as well. I.e. \(n_Q = d_{c,S} - d_{o,Q}\), \(n_S = d_{c,Q} - d_{o,S}\).

Comparing these to Corollary 5 we obtain that \(d_{c,Q} = d_{c,S}\), which is the first part of the last statement of the theorem. On the other hand \(\dim(O_Q^\perp) = n_\Theta + n_S - d_{o,Q} = n_\Theta + d_{c,S} - d_{o,S} - d_{o,Q} = n_\Theta + n_Q - d_{o,S} = \dim(O_S^\perp)\) proving the last statement of the theorem.

Now the equality in (3.48) implies that the columns of \(\begin{bmatrix} \hat{\alpha}_Q \\ \hat{\beta}_Q \end{bmatrix}\) span the un-observable subspace \(O_Q\). Thus, there exists a nonsingular matrix \(K\) for which \(\begin{bmatrix} \alpha_Q \\ \beta_Q \end{bmatrix} = \begin{bmatrix} \hat{\alpha}_Q \\ \hat{\beta}_Q \end{bmatrix} K\).

Changing the extension of \(\xi_S\) to \(\hat{R}\) to \(\hat{R} = \begin{bmatrix} \xi_S \\ K^{-1} \zeta_Q \end{bmatrix}\)

we obtain that \(\hat{R}^{-1} = [\rho_S, \alpha_Q]\) as claimed.

The statements (i) - (iv) are immediate consequences of equations (3.15), (3.30), (3.25) and (3.20) using Theorem 3.
Bibliography


