An LQG derivation of risk-neutral term structure models

Andrea Gombani* and Wolfgang J. Runggaldier †

1 Abstract

We present an alternative approach to the pricing of bonds and bond derivatives in a multivariate quadratic term structure model that is based on the solution of a linear-quadratic stochastic control problem. It leads also to an approach that is an alternative to that of computing forward prices by forward measures. We finally provide explicit formulas for the computation of bond options in a bivariate factor model.

2 Introduction

The use of the Girsanov transformation to obtain a martingale measure has become the fundamental tool of asset and bond pricing. The key feature of this technique is a change of drift which preserves trajectories. However, as is well known, this is not the unique way to change the drift of a stochastic process. In fact, the drift can also be changed by feedback, albeit with a change of the trajectories, but keeping the same measure. It turns out that in the case of a linear dynamics for the term structure, a feedback approach provides the same pricing model, which can be obtained in the traditional manner, without changing the measure at all. This is done by solving a stochastic optimal control problem. Quite surprisingly, stochastic control techniques have been adopted quite early in finance (see e.g. [6]), but never in connection with derivatives pricing.

As mentioned, the trajectories of the closed-loop model are changed with respect to those of the open loop. Nevertheless, we feel that in the bond market

*ISIB-CNR. Corso Stati Uniti 4, 35127 Padova, Italy, e-mail: gombani@isib.cnr.it
†Dipartimento di Matematica Pura ed Applicata, Università di Padova, Via Trieste 63, 35121 Padova, e-mail: runggal@math.unipd.it
this is quite irrelevant. In fact, since the observed values are eventually the rates and bond prices, it is quite indifferent, as far as pricing is concerned, whether these values are generated by an original open loop model with different trajectories (which we never observe) and the same measure, or same trajectories and a different measure. What is relevant is that they produce the same pricing model. From a computational point of view, keeping the same measure has definite advantages. A non-trivial issue where these two approaches might differ is calibration. Both calibration with respect to a different measure and closed-loop identification are difficult issues. On the last one, though, some recent progress has been made (at least for linear models, see e.g. [2]). Moreover, the fact that a solution to the Riccati equation can be easily computed makes the approach quite appealing.

In general, this approach provides a quite powerful theoretical framework, whose application goes beyond the modeling of the term structure. In fact, it can be shown that it can accommodate stock option pricing as well.

3 The arbitrage free derivation for the term structure

Consider, on a filtered probability space \((\Omega, \mathcal{F}, \mathcal{F}_t, Q)\), a model of the form

\[
\begin{align*}
\dot{x}(t) &= F x(t) dt + G dw(t) , & t \geq 0 , & x_0 = 0 \\
f(t, T) &= a(t, T) + x'(t)c(t, T)x(t)
\end{align*}
\]  

(1)

where \(x(t)\) has dimension \(n\), \(w\) is an \(m\)-dimensional Wiener process w.r. to \((Q, \mathcal{F}_t)\), the symbol \(\prime\) denotes transposition, \(F\) and \(G\) are matrices of appropriate dimensions to be considered as parameters in the model, and \(a(t, T)\) and \(c(t, T)\) are scalar and matrix-valued functions respectively, differentiable with respect to \(t\) and with \(c(t, T)\) symmetric. In (1) we consider an exponentially quadratic output model as instance of a non-affine model. By imposing the conditions of absence of arbitrage as in Proposition 3.1 below, it can be shown (see [3]) that, for a linear-Gaussian factor model, general exponentially polynomial output models with degree larger than 2 reduce to the quadratic output model, i.e. the coefficients of the powers of \(x\) larger than 2 have to be equal to zero. Notice, furthermore, that the term \(a(t, T)\) in (1) implicitly includes the observed forward rate curve \(f^*(0, T)\) for an initial time \(t = 0\). Our model (1) is thus a linear factor - nonlinear output model. Dually one could also consider nonlinear factor - affine output models and there is also some equivalence between the two possible settings.

Model (1) for the forward rates implies for the short rate \(r(t, x)\) and the zero-coupon bond prices \(p(t, T, x)\) the representations

\[
\begin{align*}
r(t, x) &= a(t) + x'(t)c(t)x(t) \\
p(t, T, x) &= \exp \left[ -A(t, T) - x'(t)C(t, T)x(t) \right]
\end{align*}
\]

(2)

with \(a(t) = a(t, t)\), \(A(t, T) = \int_t^T a(t, u) du\) and, analogously, for \(c(t, T)\). Notice that, by the above definitions, \(p(t, T, x)\) means that the bond price depends also on
the factor process $x$, evaluated at time $t$; analogously for $r(t, x)$. Although obvious in the present context, this meaning of notation will be important in the sequel (see Section 4.2).

So far $a(t, T), c(t, T)$ in (1) also appear as parameters in our model and they induce the parametric functions $a(t), A(t, T)$ and $c(t), C(t, T)$ in (2). However, in order to exclude the possibility of arbitrage, they cannot be chosen arbitrarily. We shall therefore impose on them conditions for absence of arbitrage, that can equivalently be imposed on their integrated variants $A(t, T), C(t, T)$ in (2). We have

**Proposition 3.1.** A sufficient condition for the term structure model (1), (2) to be arbitrage-free is that the coefficients $A(t, T), C(t, T)$ in (2) satisfy the system of differential equations in $t$

$$
C_t(t, T) + F^tC(t, T) + C(t, T)F - 2C(t, T)GG'C(t, T) + c(t) = 0
$$

with terminal conditions $A(T, T) = 0, C(T, T) = 0$, where $A_t(t, T)$ and $C_t(t, T)$ denote the partial derivatives with respect to $t$. The function $c(t)$ is to be considered as parameter, while for $a(t)$ we have

$$
a(t) = f^*(0, t) + \frac{1}{2} \int_0^t \beta_T(s, t)ds
$$

having put

$$
\beta(t, T) := -2 \text{tr} (G'C(t, T)G)
$$

and where $f^*(0, t)$ is the observed initial forward rate curve and the subscript $T$ denotes partial differentiation with respect to the second variable.

For the proof see [5].

Notice that, under conditions (3), the given measure $Q$ is a martingale measure for the numeraire given by the money account $B(t) := \int_0^t r(s, x)ds$.

### 4 An LQG approach to bond pricing

If a bond price is a function of a Markov process, a change of measure for the bond market will induce a change of measure on the Markov process (and conversely). Therefore, a way to compute bond prices is to change the drift in the stochastic differential equation they satisfy, so that they become the numeraire (and thus their dynamics becomes trivial) and then look at the corresponding change of drift for the underlying Markov process $x$. Obviously, once we have this information on $x$, the actual computations of the prices become very simple. The point, therefore, is to find an effective way to compute the change of drift in the underlying process $x$. It turns out that this can be done by solving a stochastic control problem. The change of this drift is obtained by state feedback and not by changing the measure (but this is irrelevant for the actual computation). This change of drift, although present, does not appear explicitly in the computation of bond prices in Section 4.1. Its role becomes apparent in the computation of forward prices in Section 4.2.
4.1 Bond prices

With bond prices described by \( p(t, T, x) \), the Term Structure Equation, under the assumption that the dynamics of \( x(t) \) is given by (1), can be written as:

\[
\frac{\partial}{\partial t} p(t, T, x) + x'(t)F' \frac{\partial}{\partial x} p(t, T, x) + \frac{1}{2} \text{tr} G^t G' \frac{\partial^2}{\partial x^2} p(t, T, x) - p(t, T, x)r(t, x) = 0
\]

with terminal condition \( p(T, T, x) = 1 \). It is well known [4] (and it can be easily verified by direct computation) that (6) can be transformed, putting

\[
W(t, T, x) := -\ln p(t, T, x)
\]

(and dropping the variables) into

\[
\frac{\partial}{\partial t} W + x'F' \frac{\partial}{\partial x} W - \frac{1}{2} \frac{\partial}{\partial x} W'GG' \frac{\partial}{\partial x} W + \frac{1}{2} \text{tr} GG' \frac{\partial^2}{\partial x^2} W + \frac{1}{2} u'u + r = 0
\]

Consider now the Hamilton-Jacobi-Bellman equation:

\[
\frac{\partial}{\partial t} W + \min_{u \in \mathbb{R}^m} \left\{ (x'F' + u'G') \cdot \frac{\partial}{\partial x} W + \frac{1}{2} \text{tr} GG' \frac{\partial^2}{\partial x^2} W + \frac{1}{2} u'u + r \right\} = 0
\]

that has as solution \( u(t, x) := -G' \frac{\partial}{\partial x} W(t, x) \) and notice that, by substituting this solution in (8), this latter equation becomes (7). On the other hand, equation (8) is the Hamilton-Jacobi-Bellman equation associated with the Linear Regulator Problem (see [1]) which can be formulated, in view of the expression (2) of \( r \), as minimizing:

\[
W(t, T, x) := \min_{u(s) \in U} \mathbb{E}_t,x \int_t^T [x'(s)c(s)x(s) + \frac{1}{2} (u'u)(s)] ds + \int_t^T a(s) ds
\]

where the expectation is with respect to the given measure \( Q \) and \( x(t) \) has dynamics:

\[
dx(t) = [Fx(t) + Gu(t)]dt + Gdw(t); \quad t \leq T
\]

and \( u(t) \) is a (feedback) control applied at time \( t \) (notice that the term involving \( a \) does not play any role in the minimization process). This problem, with final cost equal to 0 is called Lagrange problem. It is well known (see [1]) and it is not surprising that the solution is (again) given by

\[
W(t, T, x) = x'C(t, T)x + \tilde{A}(t, T) + \int_t^T a(s) ds
\]

where the functions \( C \) and \( \tilde{A} \) satisfy the equations:

\[
\frac{\partial}{\partial t} C(t, T) + C(t, T)F + F'C(t, T) - 2C(t, T)GG'C(t, T) + c(t) = 0
\]

\[
\frac{\partial}{\partial t} \tilde{A}(t, T) + \text{tr} GG'C(t, T) = 0
\]

with terminal conditions \( C(T, T) = 0 \) and \( \tilde{A}(T, T) = 0 \) for all \( T \). Notice that these equations correspond to those already derived in (3) by putting

\[
A(t, T) := \tilde{A}(t, T) + \int_t^T a(s) ds
\]
The reason for introducing this distinction between $A(t, T)$ and $\tilde{A}(t, T)$ that are related by (13) is that this allows us, as we shall show in Theorem 4.1 below, to choose the deterministic function $a$ so that, for $t = 0$, the rates coincide with the initially observed forward rates $f^*(0, t)$. This allows us to obtain models that reflect real world prices. We have now the general result.

**Theorem 4.1.** Let $r(t, x)$ be defined as in (2) and let the bond price $p(t, T, x)$ be given by

$$p(t, T, x) = E_t e^{-\int_t^T r(s, x) ds}$$

Then

$$p(t, T, x) = e^{-W(t, T, x)}$$

where $W$ is, for each $T$, of the form (11) and $a$ is determined by the boundary condition

$$W(0, T, 0) = \int_0^T f^*(0, s) ds$$

where $f^*(0, t)$ is the initially observed forward rate.

**Proof:** In view of the representations of $p(t, T, x)$ in (2) and (14) and the relationship between the equations in (3) and (12), the only thing to show is that $a(t)$ defined in the theorem satisfies (4). Since

$$W(0, T, 0) = \tilde{A}(0, T) + \int_0^T a(s) ds = \int_0^T r C(s, T) G G' ds + \int_0^T a(s) ds$$

we have, in view of (15),

$$\int_0^T r C(s, T) G G' ds + \int_0^T a(s) ds = \int_0^T f^*(0, s) ds$$

and thus, recalling (13), we can write:

$$A(t, T) = \int_t^T r G' C(s, T) G ds + \int_t^T a(s) ds$$

Differentiating now with respect to $t$ we obtain the relation (3) with $a(t)$ satisfying (4).

Notice that $A(t, T)$ has the explicit representation:

$$A(t, T) = \int_t^T r G' C(s, T) G ds + \int_t^T a(s) ds$$

$$= \int_t^T r G' C(s, T) G ds + \int_0^T a(s) ds - \int_0^t a(s) ds$$

$$= \int_t^T r G' C(s, T) G ds - \int_0^t tr G' C(s, T) G ds + \int_0^T f^*(0, s) ds$$
\[ + \int_0^t \text{tr} G' C(s, t) G ds - \int_0^t f^*(0, s) ds \]
\[ = \int_t^T f^*(0, s) ds - \int_0^t \text{tr} [G' C(s, T) G - G' C(s, t) G] ds \]

### 4.2 Forward Prices

We compute now the forward price of a bond, that is the value \( E_{t,x}^Q \tau p(\tau, T, x) \) for \( t \leq \tau \leq T \), where \( Q_\tau \) is the forward measure with respect to the numeraire \( p(t, \tau, x) \). That is, prices expressed in units of \( p(t, \tau, x) \) are \( Q_\tau \)-martingales, and so we have

\[ E_{t,x}^Q \tau p(\tau, T, x) = \frac{p(t, T, x)}{p(t, \tau, x)} \]  \hspace{1cm} (17)

Recall that in \( p(t, \tau, x) \), the dependence on the factor process \( x \) is through its value at time \( t \). We claim that we can derive this forward price as an expected value with respect to the physical measure but with a different dynamics for the factors than in (1), namely

\[ dx_\tau(t) = [(F - 2GG'C(t, \tau))x_\tau(t)]dt + Gdw(t) \]  \hspace{1cm} (18)

and that the price of a bond at time \( \tau > t \), is given by

\[ p(\tau, T, x_\tau) = e^{-x_\tau'(\tau)C(\tau, T)x_\tau(\tau) - A(\tau, T)} \]

(notice how the entire difference with respect to the change of measure approach in the above expression rests on the dynamics of \( x_\tau \), which is that of (18) instead of that of (1)).

To verify these statements, we introduce the following:

\[ p^\tau(t, T, x) := E_{t,x} p(\tau, T, x) = E_{t,x} e^{-x_\tau'(\tau)C(\tau, T)x_\tau(\tau) - A(\tau, T)} \]  \hspace{1cm} (19)

The Kolmogorov backward equation to compute this expected value is (dropping the variables):

\[ \frac{\partial}{\partial t} p^\tau + x'(F' - 2GG') \frac{\partial}{\partial x} p^\tau + \frac{1}{2} \text{tr} GG' \frac{\partial^2}{\partial x^2} p^\tau = 0 \]  \hspace{1cm} (20)

with terminal condition \( p^\tau(\tau, T, x) = e^{-x_\tau'(\tau)C(\tau, T)x_\tau(\tau) - A(\tau, T)} \). Setting

\[ W^\tau(t, T, x) := -\ln p^\tau(t, T, x) \]  \hspace{1cm} (21)

(20) becomes:

\[ \frac{\partial}{\partial t} W^\tau + x'(F' - 2GG') \frac{\partial}{\partial x} W^\tau - \frac{1}{2} \frac{\partial}{\partial x} (W^\tau)'GG' \frac{\partial}{\partial x} W^\tau + \frac{1}{2} \text{tr} GG' \frac{\partial^2}{\partial x^2} W^\tau = 0 \]  \hspace{1cm} (22)

with terminal condition \( W^\tau(\tau, T, x) = x_\tau'(\tau)C(\tau, T)x_\tau(\tau) + A(\tau, T) \). This equation is again the result of the substitution into the Hamilton-Jacobi-Bellman equation:

\[ \frac{\partial}{\partial t} W + \min_{u \in \mathbb{R}^m} \left\{ x'(F' - 2GG') + u'G' \right\} \frac{\partial}{\partial x} W + \frac{1}{2} \text{tr} GG' \frac{\partial^2}{\partial x^2} W + \frac{1}{2} u'u \right\} = 0 \]

\hspace{1cm} (23)
of \( u \) with the solution \( u(t, x) := -G' \frac{\partial}{\partial x} W^\tau(t, x) \). We therefore obtain again an LQG problem (this time in Mayer form), namely

\[
W^\tau(t, T, x) = \min_{u(s) \in U} E_{t,x} \left \{ \int_t^T \frac{1}{2} u'(s)u(s)ds + x_\tau(\tau)C(\tau, T)x_\tau(\tau) \right \} + A(\tau, T) \tag{24}
\]

So, the solution is again given by a Riccati equation, but this time an homogeneous one. Moreover, the dynamics of \( x_\tau \) is now given by

\[
dx_\tau(t) = \left[ (F - 2GG'C(t, \tau))x_\tau(t) + Gu(t) \right]dt + Gdw(t) \tag{25}
\]

that is, we modify the dynamics (18) with the addition of a control term (as was done in (10) with respect to (1)). The solution to (24) will therefore look like

\[
W^\tau(t, T, x) = x' C^\tau(t, T)x + A^\tau(t, T) \tag{26}
\]

for \( t \leq \tau \leq T \), where \( C^\tau(t, T) \) satisfies the homogenous Riccati Equation:

\[
\frac{\partial}{\partial t} C^\tau(t, T) + (F' - 2C(t, \tau)GG'C(t, \tau))C^\tau(t, T) + C^\tau(t, T)(F - 2GG'C(t, \tau)) - 2C^\tau(t, T)GG'C^\tau(t, T) = 0
\]

\[
C^\tau(\tau, T) = C(\tau, T) \tag{27}
\]

and \( A^\tau(t, T) \) satisfies the differential equation:

\[
\frac{\partial}{\partial t} A^\tau(t, T) + \text{tr} G'C^\tau(t, T)G = 0
\]

\[
A^\tau(\tau, T) = A(\tau, T) \tag{28}
\]

The last equation can be written in integral form as:

\[
A^\tau(t, T) = A(\tau, T) + \int_t^\tau \text{tr} G'C^\tau(s, T)Gds \tag{29}
\]

We are thus led to formulate the following

**Proposition 4.2.** The forward price at time \( t \) of a bond \( p(\tau, T) \) is given by:

\[
E_{t,x}^{Q_\tau} p(\tau, T, x) = p^\tau(t, T, x) \tag{30}
\]

where

\[
p^\tau(t, T, x) = E_{t,x} \exp \{-x_\tau^\prime(\tau)C(\tau, T)x_\tau(\tau) - A(\tau, T)\}
\]

\[
= \exp \{-x'(t)C^\tau(t, T)x(t) - A^\tau(t, T)\}
\]

**Proof.** First, we claim that

\[
C(t, \tau) + C^\tau(t, T) = C(t, T) \tag{31}
\]
for \( t \leq \tau \). In fact, setting \( \hat{C}(t) := C(t, \tau) + C^\tau (t, T) \) for \( t \leq \tau \),

\[
- \frac{d}{dt} \hat{C}(t) = - \frac{\partial}{\partial t} C(t, \tau) - \frac{\partial}{\partial t} C^\tau (t, T) \\
= F'C(t, \tau) + C(t, \tau)F - 2C(t, \tau)GG'C(t, \tau) + c(t) \\
+(F' - 2C(t, \tau)GG')C^\tau (t, T) + C^\tau (t, T)(F - 2GG'C(t, \tau)) \\
- 2C^\tau (t, T)GG'C^\tau (t, T) \\
= F'\hat{C}(t) + \hat{C}(t)F - 2\hat{C}(t)GG'\hat{C}(t) + c(t)
\]

which is the Riccati equation in (3) whose unique solution (with terminal condition \( C(T, T) = 0 \)) is \( C(t, T) \). Since \( \hat{C}(\tau) = C(\tau, T) \), in view of uniqueness it must be \( \hat{C}(t) = C(t, T) \) for \( t \leq \tau \), as claimed. Similarly, \( A(t, \tau) + A^\tau (t, T) = A(t, T) \). In fact, in view of (16) and (29) as well as (31),

\[
A(t, \tau) + A^\tau (t, T) = A(t, \tau) + A(\tau, T) + \int_0^\tau \text{tr } G'C^\tau(s, T)Gds + \\
\int_t^\tau f^*(0, s)ds - \int_0^t \text{tr } G'[C(s, \tau) - C(s, t)]Gds \int_t^\tau f^*(0, s)ds \\
+ \int_0^\tau \text{tr } G'[C(s, T) - C(s, \tau)]Gds + \int_t^\tau \text{tr } G'[C(s, T) - C(s, \tau)]Gds \\
= \int_0^\tau f^*(0, s)ds - \int_0^t \text{tr } G'[C(s, T) - C(s, t)]Gds = A(t, T)
\]

where in the last equality we have used (16). Therefore, recalling (13) and (26) with (31), we immediately have

\[
p(t, \tau, x)p^\tau (t, T, x) \\
= \exp\{-x'(t)C(t, \tau)x(t) - A(t, \tau)\} \cdot \exp\{-x'(t)C^\tau (t, T)x(t) - A^\tau (t, T)\} \\
= \exp\{-x'(t)C(t, T)x(t) - A(t, T)\} = p(t, T, x)
\]

that is, in view of (17),

\[
p^\tau (t, T, x) = E_t^Q p(\tau, T, x)
\]

which proves (30).

4.3 Forward Measure

The question which naturally arises now is the connection with the usual forward measure which is normally used to compute (17). In fact, it turns out that the two tools are equivalent.

- Let \( x(\tau) \) be the value in \( \tau \) of the solution to (1) with initial condition \( x(t) = x \).
- Let \( x^\tau (\tau) \) be the value in \( \tau \) of the solution to (18) with initial condition \( x(t) = x \).
**Proposition 4.3.** Given \( \tau \), the two random variables \( x(\tau) \) and \( x_\tau(\tau) \) have the same Gaussian distribution, the first under the forward measure \( Q_\tau \) (with numeraire \( p(t, \tau, x) \)), the second under the standard martingale measure \( Q \) (with numeraire \( B_t \)).

**Proof.** : For the numeraire \( p(t, \tau, x) \) we have, under \( Q \),
\[
dp(t, \tau, x) = p(t, \tau, x) \left[ r(t, x) dt - 2x'(t) C(t, \tau) G dw_t \right]
\]
(32)

For the Radon-Nikodym derivative \( L_t = \frac{p(t, \tau, x)}{B_t} \) one then has
\[
dL_t = - \left[ 2x'(t) C(t, \tau) \right] G L_t dw_t
\]
(33)

It follows that, denoting by \( w^\tau_t \) a Wiener process under \( Q_\tau \),
\[
dw^\tau_t = dw_t + 2G'C(t, \tau)x(t) dt
\]
(34)

For \( x(t) \) satisfying (1) under \( Q \) one then has, under \( Q_\tau \),
\[
dx(t) = \left[ (F - 2GG'C(t, \tau))x(t) \right] dt + G dw^\tau_t
\]
(35)

Since the dynamics in (35) is identical to those in (18) and \( x(t) = x \) in both cases, the distribution of \( x(\tau) \) under \( Q_\tau \) and \( x_\tau(\tau) \) under \( Q \) are the same and given by a Gaussian.

**Remark 4.4.** An alternative approach to obtain the same result as in the previous Proposition could be to show that from the following equality, that derives from (30), namely
\[
E^{Q_\tau}_{t,x} \left\{ \exp \left[ -x'(\tau)C(\tau, T)x(\tau) - A(\tau, T) \right] \right\}
\]
(36)

follows the equality of the two Gaussian distributions, that of \( x(\tau) \) under \( Q_\tau \) and that of \( x_\tau(\tau) \) under \( Q \), given that \( x(t) = x_\tau(t) = x \).

We have now the following result, which generalizes the scalar result of [7]:

**Proposition 4.5.** Given a maturity \( \tau \) and an integrable claim \( H(x(\tau)) \), its arbitrage free price at \( t < \tau \) is
\[
\Pi_t = E_{t,x} \left\{ e^{-\int_t^{\tau} r(s,x) ds} H(x(\tau)) \right\}
\]
(37)

\[
= p(t, \tau, x) E^{Q_\tau}_{t,x} \left\{ H(x(\tau)) \right\} = e^{-W(t, \tau, x)} E_{t,x} \left\{ H(x_\tau(\tau)) \right\}
\]

**Proof.** : The first equality follows from the definition of \( Q \) as martingale measure for the numeraire \( B_t \), the second from the definition of the forward measure \( Q_\tau \) and the third follows from (14) and the equality of the distributions of \( x(\tau) \) under \( Q_\tau \) and of \( x_\tau(\tau) \) under \( Q \).
4.4 Pricing of a bond derivative

We derive now an explicit formula for the pricing of a bond option that is based on (37) and on the representation of the factor process \( x_\tau(\cdot) \) in section 4.2. First we have

Remark 4.6. If \( \Phi_\tau(\tau,t) \) denotes the fundamental solution of (18), we immediately see that, for \( \tau > t \) the conditional mean of the Gaussian process \( x_\tau(\cdot) \) given \( x_\tau(t) = x \) is expressed by:

\[
E_{t,x} x_\tau(\tau) = \Phi_\tau(\tau,t)x
\]

and its conditional variance by

\[
E_{t,x} [x_\tau(\tau) - E_{t,x} x_\tau(\tau)][x'_\tau(\tau) - E_{t,x} x'_\tau(\tau)] = E \int_t^\tau \Phi_\tau(\tau,s) G dw(s) dw(s)' G' \Phi_\tau(\tau,s)' ds
\]

In view of the previous section, we can express the arbitrage free price of a claim by formula (37) where now \( x_\tau(\cdot) \) is the Gaussian process with mean \( \mu \) given by (38) and variance \( \Sigma \) given by (39). In particular, the value of a call option with strike price \( K \) and maturity \( \tau \) on a bond with maturity \( T \) will be

\[
\Pi_t = e^{-W(t,\tau,x)} E_{t,x} \max \{ 0, p(\tau,T,x) - K \} = e^{-W(t,\tau,x)} E_{t,x} \max \{ 0, e^{-W(\tau,T,x)} - K \}
\]

where

\[
E_{t,x} \max \{ 0, e^{-W(\tau,T,x)} - K \} = \frac{1}{\sqrt{(2\pi)^n \det \Sigma}} \int_{-W(\tau,T,x) > -\ln K} e^{-\frac{1}{2} (\xi - \mu)' \Sigma^{-1} (\xi - \mu)} e^{-\xi'C(\tau,T)\xi - A(\tau,T)} d\xi_1...d\xi_n
\]

\[
- \frac{K}{\sqrt{(2\pi)^n \det \Sigma}} \int_{-W(\tau,T,x) > -\ln K} e^{-\frac{1}{2} (\xi - \mu)' \Sigma^{-1} (\xi - \mu)} d\xi_1...d\xi_n
\]

Assume now that \( \Sigma \) has full rank (it can be shown that this is always the case if the pair \( (F,G) \) is controllable, that is if the matrix \( G, FG, ... F^{n-1}G \) has full column rank). We perform then a suitable change of variables remembering that: a) two positive definite matrices can be simultaneously diagonalized by congruence and the transformation can be chosen so that one is the identity; b) if \( \Sigma \) is positive definite, we can complete the squares. The above integrals can then be rewritten as

\[
E_{x,t} \max \{ 0, e^{-W(\tau,T,x)} - K \} = \lambda \int_{\sum_{i=1}^n \gamma_i (\xi_i - \mu_i)^2 + \alpha \leq 0} e^{-\frac{1}{2} (\xi_1 + \xi_2 + ... + \xi_n) d\xi_1...d\xi_n} - \eta \int_{\sum_{i=1}^n \delta_i (\xi_i - \nu_i)^2 + \beta \leq 0} e^{-\frac{1}{2} (\xi_1 + \xi_2 + ... + \xi_n) d\xi_1...d\xi_n}
\]

for suitable constants \( \gamma_i, \delta_i, \mu_i, \nu_i, \alpha, \beta, \lambda \) and \( \eta \); so these integrals are each the (gaussian) measure of an ellipsoid. Thus, in principle, derivative prices can be computed.
in a standard manner. Nevertheless, these are multiple integrals and thus their actual computation is quite demanding. In the case of \( n = 2 \), though, the formulas, although complicated, can be reduced to calculating the value of two single integrals and require therefore a computational effort comparable with that of Black and Scholes. In fact, using a standard trick in the first integral, that is the substitutions \( \xi_1 = \rho \cos \theta \) and \( \xi_2 = \rho \sin \theta \), we see that the set \( D_{\xi_1, \xi_2} := \{(\xi_1, \xi_2); \gamma_1(\xi_1 - \mu_1)^2 + \gamma_2(\xi_2 - \mu_2)^2 + \alpha \leq 0 \} \) is mapped into the set
\[
D_{\rho, \theta} := \{(\rho, \theta); p(\rho, \theta) \leq 0 \}
\]
where we have set (using the above substitution):
\[
p(\rho, \theta) := \gamma_1(\rho \cos \theta - \mu_1)^2 + \gamma_2(\rho \sin \theta - \mu_2)^2 + \alpha
\]
\[
= \rho^2(\gamma_1 \cos^2 \theta + \gamma_2 \sin^2 \theta) - \rho(2\gamma_1 \mu_1 \cos \theta + 2\gamma_2 \mu_2 \sin \theta) + \gamma_1 \mu_1^2 + \gamma_2 \mu_2^2 + \alpha
\]

As a function of \( \rho \), the above is a second degree polynomial with roots \( \rho_1(\theta), \rho_2(\theta) \) in \( D_{\rho, \theta} \). Notice that, if \( p(\rho, \theta) = 0 \), also \( p(-\rho, \theta + \pi) = 0 \). Therefore the set \( \Gamma \) of points satisfying (43) is constituted by two ellipses (as long as the discriminant \( \Delta \) of (43) is non negative for some \( \theta \)), and in fact we can choose the one denoted by \( \Gamma_+ \) such that \( \rho \geq 0 \); thus, there are two possibilities:

a) The ellipse \( \Gamma_+ \) does not contain the origin so, if they are real, both roots \( \rho_1(\theta), \rho_2(\theta) \) have the same sign. Therefore, denoting by \( I_\theta \) the subinterval of \([0, 2\pi]\) where the discriminant \( \Delta \) of (43) is non negative and \( \rho \geq 0 \), both roots are positive: if we make the convention that \( \rho_1(\theta) \leq \rho_2(\theta) \), we can write (42) as:
\[
\int_{D_{\xi_1, \xi_2}} e^{-\frac{1}{4}(\xi_1^2 + \xi_2^2)} d\xi_1 d\xi_2 = \int_{D_{\rho, \theta}} e^{-\frac{1}{4}\rho^2} \rho d\rho d\theta = -\int_{I_\theta} e^{-\frac{1}{4}\rho^2} \left[ \rho(\rho_2(\theta)) - \rho(\rho_1(\theta)) \right] d\theta
\]
\[
= \int_{I_\theta} \left[ e^{-\frac{1}{4}\rho_1^2(\theta)} - e^{-\frac{1}{4}\rho_2^2(\theta)} \right] d\theta
\]

b) The ellipse \( \Gamma_+ \) contains the origin, and thus there are two real roots for each \( \theta \in [0, 2\pi] \) and the roots have opposite sign. Denoting the positive root by \( \rho_2(\theta) \), we can write (42)
\[
\int_{D_{\xi_1, \xi_2}} e^{-\frac{1}{4}(\xi_1^2 + \xi_2^2)} d\xi_1 d\xi_2 = \int_{D_{\rho, \theta}} e^{-\frac{1}{4}\rho^2} \rho d\rho d\theta = -\int_{0}^{2\pi} e^{-\frac{1}{4}\rho^2} \left[ \rho(\rho_2(\theta)) \right] d\theta
\]
\[
= \int_{0}^{2\pi} \left[ 1 - e^{-\frac{1}{4}\rho_2^2(\theta)} \right] d\theta
\]

In both cases, this is a simple integral whose value is easily computed numerically. A similar procedure can be employed for the second integral in (42).
Bibliography


