ON EXTERNAL SPECTRAL FACTORS
AND GEOMETRIC CONTROL

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Abstract

We investigate here how the geometric control theory of Basile, Marro and Wonham can be obtained in a Hilbert space context, as the byproduct of the factorization of a spectral density with no zeros on the imaginary axis. We show how stabilizable and controllability subspaces can be obtained as images of orthogonal projections of coinvariant subspaces onto a semimartingale (markovian) subspace of the Hardy space of square integrable functions analytic in the right half plane.

1 Introduction

The problem we study in this paper has a long history, motivated both by spectral factorization and control problems (they both end up with a Riccati equation). The approach we take is nevertheless new: we exploit the Hilbert space structure (as is done in classical stochastic realization theory), but instead of analyzing the geometry in the stochastic domain, we carry this structure entirely into the frequency setting. The main advantage is a computational one. In fact, we reduce the problem to the study of a set of stable all-pass functions. In particular, in this paper we investigate the relations between minimal spectral factors (in this context) and the geometric control theory developed by Basile and Marro [1] and Wonham [12]. The result is that there is no stochastics in this paper, but just Hilbert spaces and geometric control theory. It is nevertheless only in stochastic realization theory (see [9], [10] and [6]) that the factorizations we will investigate and determine a maximal antistabilizable subspace \( V \) and a maximal stabilizable subspace \( \overline{V} \) of \( \Phi \) which are output nulling. Moreover, the supremal output nulling controllability subspace \( R \) is the intersection of \( V \) and \( \overline{V} \). The step from \( Q' \) to the supremal antistabilizable subspace is quite simple. Let \( WK \) be the Douglas-Shapiro-Shields factorization (see below) of \( W \); then \( V \) is (isomorphic to) the projection of the coinvariant subspace \( H(Q') \) onto \( H(K) \). The coinvariant subspace \( H(K) \) is quite important in stochastic realization theory (see e.g. [9] or [11]) because it represents the state space for the realization of \( W \).

One advantage of this approach is computational. The factorizations and the projections can all be computed using state space formulas and solving Lyapunov and Riccati equations. This should not only allow for a simple algorithm for determining the controlled invariant subspaces, but also, through a suitable choice of basis which exploits SVD techniques on the solutions to the Lyapunov equations for the projections, to a solution for approximate problems.

Our work has been largely inspired by [10]: we believe that the work exposed is a natural extension of the results of that paper in the context of the geometric stochastic realization theory of Lindquist and Picci.

2 Preliminaries and notation

We work in the Hilbert space setting of the plane; we define \( L_2^2(\mathbb{I}) \) to be the set of the vector or matrix valued (the proper dimension will be clear from the context) square integrable functions on the imaginary axis, and \( H_+^2 \) to be the subspace of \( L_2^2 \) of functions analytic in the right half-plane and such that

\[
\sup_{x > 0} \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{tr}(F^*(x + iy)F(x + iy)) \, dy < \infty
\]

where * denotes transposed conjugate. If \( F \) and \( G \) are column vectors, the inner product in \( H_+^2 \) is

\[
(F, G) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G^*(i\omega) F(i\omega) \, d\omega
\]

Analogously, \( H_+^\infty \) is the subspace of \( L_2^\infty \) of functions analytic in the right half-plane and such that

\[
\left(\begin{array}{cc}
A & B \\
C & D
\end{array}\right)
\]

be a fixed realization of \( W \): we show here how the functions \( Q' \) and \( Q'' \) associated to \( W \) determine a maximal antistabilizable subspace \( V \) and a maximal stabilizable subspace \( \overline{V} \) of \( \Phi \) which are output nulling. Moreover, the supremal output nulling controllability subspace \( R \) is the intersection of \( V \) and \( \overline{V} \). The step from \( Q' \) to the supremal antistabilizable subspace is quite simple. Let \( WK \) be the Douglas-Shapiro-Shields factorization (see below) of \( W \); then \( V \) is (isomorphic to) the projection of the coinvariant subspace \( H(Q') \) onto \( H(K) \). The coinvariant subspace \( H(K) \) is quite important in stochastic realization theory (see e.g. [9] or [11]) because it represents the state space for the realization of \( W \).

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Let \( F \) be a function of \( L^2 \): we denote by \( P^+ F (P^- F) \) the orthogonal projection of \( L^2 \) onto \( H^2_+ (H^2_-) \).

A function \( Q \in H^\infty_+ \) is inner if it is square and \( Q^* Q = QQ^* = I \). It is well known that a column vector space \( M \in H^2_+ \) is invariant under multiplication by \( e^{-it} \) for \( t \geq 0 \) if and only if it is of the form \( M = QH^2_+ \) (Beurling’s Theorem). Similarly, a row space \( N \) is invariant if and only if it is of the form \( N = H^2_- Q \). We set \( H_+(Q) := (QH^2_+)^+ \) and \( H_-(Q) := (H^2_- Q)^+ \). Clearly it is \( P^+ e^{it} H_+(Q) \subset H_+(Q) \) (similarly for \( H_-(Q) \)).

Remark: since we will mostly use row vectors in \( H^2_+ \), we will write \( H(K) \) for \( H_+(K) \) when there is no ambiguity. A full column-rank \( p \times m_0 \) rational matrix function \( G \in H^\infty_+ \) is said to be minimum-phase or outer (on the right) if rank \( G(s) = m_0 \) for \( \Re s > 0 \). It’s well known that a rational function \( F \) in \( H^\infty_+ \) admits an outer inner factorization

\[
F = F_0 Q
\]

This factorization is unique up to a unitary constant matrix. The factorization \( W = WK \) of \( W \) in \( H^2_+ \) is a Douglas-Shapiro-Shields (DSS) factorization of \( W \) if \( W \in H^2_+ \), \( K \) is inner and the degree of \( K \) is as small as possible. Also this factorization is unique up to a unitary constant matrix. A matrix function \( W \) is said to be maximum-phase if its DSS factor \( W \) is (conjugate) outer in \( H^2_- \). Consequently \( W_+ \) is the maximum-phase spectral factor of a spectral density \( \Phi \) if the DSS factor \( W_+ \) of \( W_+ \) is minimum-phase in \( H^2_+ \). Given two inner functions \( P \) and \( Q \) we say that \( \overline{QP} \) is a skew-prime factorization (see [4]) if \( P \bar{Q} = \bar{Q} P \), \( P \) and \( Q \) are left coprime and \( P \) and \( Q \) are right coprime.

From now on we assume all the functions to be rational. The notation \( W = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \) indicates that the quadruple \( (A, B, C, D) \) is a minimal realization of \( W \). By \( A^\# \) we denote the Moore-Penrose inverse of a matrix \( A \).

The following is a well known characterization of inner functions, see [2], [3] or Fuhrmann and Ober [7].

**Proposition 2.1**

1. The square proper stable rational matrix function \( \hat{R} \) is inner if and only if it has a minimal realization

\[
\hat{R} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}
\]

where \( DD^* = I \) and \( P \) solves the Lyapunov equation

\[
AP + PA^* + BB^* = 0.
\]

2. Similarly \( \hat{R} \) is inner if and only if it has a minimal realization

\[
\hat{R} = \begin{pmatrix} A \\ C \end{pmatrix} \begin{pmatrix} Q^* \\ D \end{pmatrix}
\]

where \( D^* D = I \) and \( Q \) solves the Lyapunov equation

\[
A^* Q + QA + C^* C = 0.
\]

We define the multiplicity of an \( m \times m \) rational inner function \( K = \begin{pmatrix} A & B \\ -B^* P^{-1} & I \end{pmatrix} \) as the rank of the matrix \( B \).

**Corollary 2.1** Let \( K = \begin{pmatrix} A & B \\ -B^* P^{-1} & I \end{pmatrix} \) be an \( m \times m \) inner matrix of multiplicity \( m_0 \); then there exists a unitary matrix \( V \) such that \( KV = \begin{pmatrix} K_0 & 0 \\ 0 & I \end{pmatrix} \)

and \( K_0 \) is \( m_0 \times m_0 \).

**Lemma 2.1**

\[
S_i = \begin{pmatrix} A_i & B_i \\ C_i & D_i \end{pmatrix} \quad i = 1, 2
\]

be minimal realizations of two inner functions. Let

\[
A_e := \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}, \quad B_e := \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}.
\]

Suppose \( S_1 \) and \( S_2 \) are right coprime and define \( S_L = S_1 \cup S_2 \) to be the least common left multiple of \( S_1 \) and \( S_2 \), and let \( \overline{S_2} := S_L S_1^* \). Then

\[
P_{R_i(\overline{S_2})} S_i (s I - A_2)^{-1} B_2 = (s I - A_2)^{-1} B_2 S_1(s)
\]

where \( B_2 := -P_{11} P_{12}^{-1} B_1 + B_2 \) and let \( P_e = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} \) is the solution to the Lyapunov equation

\[
A_e P_e + P_e A_e^* + B_e B_e^* = 0
\]

3 Geometric control

We start with a theorem which characterizes all minimal stable spectral factors of a given dimension of a rational \( p \times p \) spectral density \( \Phi \) of rank \( m_0 \) (see [10] and [6]). Let \( W_0 \) and \( W \) be the \( p \times m_0 \) minimum-phase and maximum-phase spectral factors of \( W \).

**Theorem 3.1** Let \( W \) be a \( p \times m \) spectral factor for the spectral density \( \Phi \) with rank \( m_0 \leq m \) and no rank drop at infinity. Then there exist inner functions \( Q' \) and \( Q'' \), essentially unique up to a constant unitary matrix factor, such that

\[
W = [W_-, 0] Q' = [W_+, 0] Q''
\]

We will need a simple Lemma about DSS factorizations (see e.g. [7]).

**Lemma 3.1** Let \( \overline{W} = WK^* \) be the DDS factorization of \( W \) over \( H^\infty_+ \) and let \( W = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \), \( \overline{W} = \begin{pmatrix} \overline{A} & \overline{B} \\ \overline{C} & \overline{D} \end{pmatrix} \) and \( K = \begin{pmatrix} A & B \\ -B^* P^{-1} & I \end{pmatrix} \) be minimal realizations, with \( P \) satisfying \( AP + PA^* + BB^* = 0 \). Then

\[
\overline{W} = \begin{pmatrix} -A^* \\ DB^* + CP \end{pmatrix} \begin{pmatrix} P^{-1} B \\ D \end{pmatrix}
\]
and

\[ W = \begin{pmatrix} -A^* -DB + CP^{-1} & PB \end{pmatrix} \]

**Lemma 3.2** Let \( S_i = \begin{pmatrix} A_i & B_i \\ -B_i^* P^{-1} & I \end{pmatrix} \), \( i = 1, 2 \), be minimal realizations of inner functions of degree \( n_1 \) and \( n_2 \) respectively.

1. Let \( P_{12} \) be the unique solution to the equation

\[ A_1 P_{12} + P_{12} A_2^* + B_1 B_2^* = 0 \]

Then

\[ P_{H_i(S_2)} (sI - A_1)^{-1} B_1 = P_{12} P_{22}^{-1} (sI - A_2)^{-1} B_2 \]

\[ \text{(8)} \]

2. Similarly, let \( Q_{21} \) be the solution to the equation

\[ A_2^* Q_{21} + Q_{21} A_1 + P_{21}^{-1} B_2 B_1^* P_{11}^{-1} = 0 \]

Then

\[ P_{H_i(S_2)} B_1 P_{11}^{-1} (sI - A_1)^{-1} = B_2 P_{21}^{-1} (sI - A_2)^{-1} P_{22} Q_{21} \]

\[ \text{(9)} \]

\[ \text{(10)} \]

**Proof:** We set

\[ A_e := \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}, \quad B_e := \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \]

It is then immediate to check that \( P_e = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} \) is the solution to the Lyapunov equation

\[ A_e P_e + P_e A_e^* + B_e B_e^* = 0 \]

\[ \text{(11)} \]

So, set \( u_i = e_i (sI - A_1)^{-1} B_1 \) for \( i = 1, \ldots, n_1 \) and \( v_j = e_j (sI - A_2)^{-1} B_2 \) for \( j = 1, \ldots, n_2 \), where \( e_i \) and \( e_j \) are the standard right unit vectors of dimensions \( n_1 \) and \( n_2 \). It’s quite well known (see e.g. [2]) that

\[ \langle u_i, u_j \rangle = [P_{11}]_{ij}, \quad \langle v_i, v_j \rangle = [P_{22}]_{ij} \]

We decompose now the vector \( u_i = e_i (sI - A_1)^{-1} B_1 \) into its components in \( H_e(S_2) \) and its orthogonal complement. Thus we write

\[ u_i = \sum_{k=1}^n c_{ik} v_k + \left( u_i - \sum_{k=1}^n c_{ik} v_k \right), \]

with the coefficients determined by the orthogonality relations \( \langle u_i, v_k \rangle = 0 \). Thus we obtain \( \langle u_i, v_j \rangle = \sum_{k=1}^n c_{ik} v_k \langle v_j, v_k \rangle \), which is equivalent to \( P_{12} = C P_{22} \). Thus \( C = P_{12} P_{22}^{-1} \). This proves (8). The dual equation is derived similarly.

We want now to provide a Hilbert space characterization of controlled invariant subspaces. Let \((A, B)\) be a controllable pair. A subspace \( \mathcal{V} \) of \( \mathbb{C}^n \) is said to be controlled invariant if there exists a feedback matrix \( F \) such that

\[ (A + BF) \mathcal{V} \subset \mathcal{V} \]

We need to translate vectors on \( \mathbb{C}^n \) into object of \( H^+_2 \). This is done by means of the following map. Let

\[ K = \begin{pmatrix} A \\ -B^* P^{-1} \end{pmatrix} \]

be assigned; we define the map

\[ I_{A,B}: \mathbb{C}^n \mapsto H(K) \]

\[ I_{A,B} x = \xi^* P^{-1} (sI - A)^{-1} B \]

For any subspace \( \mathcal{V} \) in \( \mathbb{R}^n \), we can then define its image in \( H^+_2 \) as the subspace

\[ X_{\mathcal{V}} := I_{A,B} \mathcal{V} \]

We say that the controlled invariant subspace \( \mathcal{V} \) is antistabilizable if there exists a feedback \( F \) such that \((A + BF) \mathcal{V}\) is antistabilizable. Similarly we say that \( \mathcal{V} \) is stabilizable if there exists a feedback \( F \) such that \((A + BF) \mathcal{V}\) is stable. A subspace \( \mathcal{V} \) is output nulling if \((A + BK) \mathcal{V} \subset \mathcal{V} \subset \ker(C + DF)\) for some feedback matrix \( F \). Let \( \mathcal{X} \subset \mathbb{C}^n \); by the notation \((A, \mathcal{X})\) we indicate the subspace span\{\(A^n X; 0 \leq k \leq n\}\}. A subspace \( \mathcal{R} \subset \mathbb{C}^n \) is a controllability subspace if there exist a feedback matrix \( F \) and a matrix \( G \) such that

\[ \mathcal{R} = (A + BF) \text{Im} BG \]

A subspace \( \mathcal{R} \subset \mathcal{V} \) is a supremal controllability subspace in \( \mathcal{V} \) if it is not properly contained in any other controllability subspace of \( \mathcal{V} \). It is well known (see [12]) that this space is unique. Let therefore \( \mathcal{R} \) be the supremal controllability subspace of \( \mathcal{V} \), and set \( \mathcal{V}^0 := \mathcal{V}/\mathcal{R} \). It is well known (see [12]) that \( \mathcal{V} \) is antistabilizable if and only if \((A + BF) \mathcal{V}^0 \) is antistabilizable.

**Lemma 3.3** Let \((A, B)\) be a controllable pair, and \( \mathcal{V} \) an controlled invariant subspaces of \( \mathbb{C}^n \); set \( K = \begin{pmatrix} A \\ -B^* P^{-1} \end{pmatrix} \) to be the inner matrix associated to \((A, B)\).

1. The controlled invariant subspace \( \mathcal{V} \) is antistabilizable if and only if there exists a feedback Q' such that

\[ X_{\mathcal{V}} := P_{H(K)} H(Q') \]

2. The controlled invariant subspace \( \overline{\mathcal{V}} \) is stabilizable if and only if there exists an inner function \( Q'' \) left coprime with \( K \) such that

\[ X_{\overline{\mathcal{V}}} := P_{H(K)Q''} H(Q'') K_+ \]

where \( Q'' \) and \( K_+ \) are uniquely determined by the relation

\[ KQ'' = \overline{Q''} K_+ \]

**Proof:**

1. We first show that if \( X_{\mathcal{V}} \) is as above, then \( \mathcal{V} \) is antistabilizable \((A, B) - \text{ invariant})\). Let \( Q' = \begin{pmatrix} A \\ -B^* P^{-1} \end{pmatrix} \) be a minimal realizations of the \( m \times m \) inner functions \( Q' \). Let

\[ A_e := \begin{pmatrix} A \\ 0 \end{pmatrix}, \quad B_e := \begin{pmatrix} B \\ 0 \end{pmatrix} \]
and let \( P_e = \left( \begin{array}{c} P' \\ P_{21} \\ P \\ P_e \\ H \end{array} \right) \) be the solution to the Lyapunov equation

\[
A_e P_e + P_e A_e' + B_e B_e' = 0
\]  
(12)

we divide the proof into steps.

(a) We claim that the rows of

\[
P_{H(K)}(sI - A)^{-1}B = P_{H(K)}(sI - A)^{-1}B
\]

(the equality is true after Lemma 3.2) span an invariant subspace for \( A_1 := P_{H(K)}^H A_1 P_{H(K)} \).

In fact, we can write

\[
P_{H(K)}^H A_1 P_{H(K)}^{-1}(sI - A)^{-1}B \subset P_{H(K)} H(Q'')
\]

since clearly

\[
\text{span}\{P_{H(K)}^{-1}(sI - A)^{-1}B\} = \text{span}\{P_{H(K)}^H P_{H(K)}^{-1}(sI - A)^{-1}B\}
\]

(b) Let now \( V := \text{Im} P_{21} \). Then it is true that

\[
V = \{ \xi : \xi^* P_{H(K)}^{-1}(sI - A)^{-1}B \in P_{H(K)} H(Q'') \}
\]

In fact, suppose \( \xi \) is such that \( \xi^* P_{H(K)}^{-1}(sI - A)^{-1}B \subset P_{H(K)} H(Q'') \). Since a basis for \( P_{H(K)} H(Q'') \) is given by the rows of \( P_{H(K)}^{-1}(sI - A)^{-1}B \) it is necessarily \( \xi \in \text{Im} P_{21} \). The converse is obvious.

(c) We have

\[
\hat{A}_{1|V} = -(A + BB^* P_{21}^H)|_V
\]

The element in the lower right corner of (12) yields

\[
AP_{21} + P_{21} A^* + BB^* = 0
\]

or

\[
AP_{21} P_{21}^H + P_{21} A^* P_{21}^H + BB^* P_{21} = 0
\]

Thus we can write

\[
P_{21} A^* P_{21}^H|_V = -(A + BB^* P_{21}^H)|_V
\]

that is, \( V \) is \((A, B)\)-invariant.

Suppose now \( V \) is antistabilizable controlled invariant; then there exists a feedback matrix \( F \) such that \((A + BF)|_V\) is antistable, and \((A + BF)|_V \subset V\). Let \( P_{21} \) be any full column-rank matrix having image \( V \). Since \( P_{21} P_{21}^H \) is a projection onto \( V \),

\[
(A + BF)|_V = P_{21} P_{21}^H (A + BF) P_{21} P_{21}^H
\]

therefore we can define

\[
A^* := -P_{21}^H (A + BF) P_{21}
\]

and \( B = P_{21}^* F^* \). Define \( Q' = \left( \begin{array}{c} A \\ -B^* P_{21}^{-1} \\ B \end{array} \right) \).

A simple computation yields

\[
AP_{21} + P_{21} A^* + BB^* = 0
\]

which is exactly the lower left block of the Lyapunov equation (11) associated to the projection of \( H(Q') \) onto \( H(K) \) (cfr. Lemma 3.2).  

2. The proof is obtained by duality considerations.

\[\square\]

Lemma 3.4 Let \( W = \left( \begin{array}{c} A \\ B \\ C \\ D \end{array} \right) \) be a minimal realization of the spectral factor \( H(K) \).

1. Let \( V \) be a controlled invariant subspace, and let \( Q' = \left( \begin{array}{c} A \\ -B^* P_{21}^{-1} \\ B \end{array} \right) \) be an inner function such that \( X_V = P_{H(K)}^H H(Q') \). Then \( W(Q')^* \) is stable if and only if

\[
V \subset \ker(C + DB^* P_{21}^H)
\]

where \( P_{21} \) satisfies the equation

\[
AP_{21} + P_{21} A^* + BB^* = 0
\]

2. Let \( \overline{A} \) be an controlled invariant subspace and let \( \overline{Q}'' = \left( \begin{array}{c} A \\ -B^* P_{21}^{-1} \\ B \end{array} \right) \) be a minimal realizations of the \( m \times m \) inner functions \( \overline{Q}'' \) such that \( X_{\overline{V}} = P_{H(K)}^H H(\overline{Q}'') K_+ \) then \( \overline{W} \overline{Q}'' \) is \( H^\infty \) if and only if

\[
V \subset \ker(C - D(B^* P_{21}^H P_{21}^{-1} - B^* P_{21}^{-1}))
\]

where \( \overline{Q}_{21} \) satisfies

\[
A^* \overline{Q}_{21} + \overline{Q}_{21} A + P_{21}^{-1} BB^* = 0
\]

Proof:

1. Let \( u = \zeta(sI - A)^{-1} B; \) then \( W u^* \) is analytic in the positive closed halfplane \( C^+ \) and so

\[
\int_{\Gamma} W(s) u^*(s) ds = 0
\]

for any closed curve \( \Gamma \) contained in \( C^+ \); in particular,

\[
\int_{\Gamma_1} Du^*(s) ds + \int_{\Gamma_2} C(sI - A)^{-1} Bu^*(s) ds = 0
\]

for any pair of closed curves \( \Gamma_1 \) and \( \Gamma_2 \) containing the poles of \( u^* \). Now it is
\[ \frac{1}{2\pi i} \int_{\Gamma_1} Du^*(s) ds \]
\[ = \frac{1}{2\pi i} \int_{\Gamma_2} DB^*(-sI - A)^{-1} \zeta ds = -2\pi i DB^\ast \zeta \]

since \( \int_{\Gamma_1} (sI + A)\zeta ds = 2\pi i I \). The second integral is easily computed observing that, since the two functions are strictly proper, we can integrate along the imaginary axis (with a change of sign, since we are reversing the pattern of integration); but we obtain, in this manner the inner product matrix of the basis \((-sI - A)^{-1}B\) and \((-sI - A)^{-1}B\). We can therefore substitute to \((-sI - A)^{-1}B\) its projection onto \(H(K)\). Thus we obtain

\[ -\frac{1}{2\pi i} \int_{\Gamma_2} C(sI - A)^{-1}Bu^*(s) ds \]
\[ = \frac{1}{2\pi i} \int_{\Gamma_2} C(sI - A)^{-1}BB^*(sI - A)^{-1} \zeta ds \]
\[ = \int C(sI - A)^{-1}BB^*(-sI - A)^{-1}P_{21} \zeta ds \]
\[ = 2\pi i CP_{21} \zeta \]

Putting things together we obtain \((DB^* + CP_{21}) \zeta = 0\) and remembering that \(\xi \in \mathcal{V}\) if and only if there is a \(\zeta\) for which \(\xi = P_{21} \zeta\) we can eventually write

\[ (DB^*P_{21} + C) \xi = 0 \quad \xi \in \mathcal{V} \quad (15) \]

Conversely, suppose \(\mathcal{V}\) is a controlled invariant subspace, and that \(Q' = \left( \begin{array}{cc} A & B \\ -B^\ast P^{-1} & I \end{array} \right)\) is an inner function associated to \(\mathcal{V}\) such that (13) is satisfied; then, by backtracking the above argument, we obtain \(\int W(s)u^*(s) ds = 0\) for each \(u \in H_r(Q')\). Since \(H_r(Q')\) is coinvariant, this implies that \(W \in H^2_{\mathcal{V}} Q'\), as wanted.

2. The proof is derived by duality considerations and is omitted.

**Theorem 3.2** Let \(W = \left( \begin{array}{cc} A & B \\ C & D \end{array} \right)\) and let \(Q'\) and \(Q''\) be as in (7). Then the maximal antistabilizable controlled invariant output nulling subspace is

\[ \mathcal{V} := \{ \xi; \xi^* P^{-1}(sI - A)^{-1}B \subset P_{H(K)} H(Q') \} \]

and the maximal stabilizable output nulling controlled invariant subspace is

\[ \mathcal{V} := \{ \xi; \xi^* P^{-1}(sI - A)^{-1}BQ'' \subset P_{H(K)Q''} H(Q')K_+ \} \]

**Proof:** In view of Lemma 3.3, \(\mathcal{V}\) is an antistabilizable controlled invariant subspace, and in view of Lemma 3.4, it is an output nulling subspace. Suppose \(W = W_1 Q_1\), and denote by \(V_1\) the controlled invariant output nulling subspace associated to \(Q_1\); since \(Q'\) is the inner factor of \(W_1, Q_1 \mid pQ',\) and therefore it’s not difficult to see that \(V_1 \subset \mathcal{V}\). The proof of the other statement is similar.

**Corollary 3.1** Let \(W = \left( \begin{array}{cc} A & B \\ C & D \end{array} \right)\), with \(D\) full column rank. Then

\[ (A + BB^*P_{21}) | \mathcal{V} = (A - BD^{-L}C) | \mathcal{V} \]

and

\[ (A + B(B^* - B^* P_{21}^{-1} Q_{21}^{-1}) P^{-1}) | \mathcal{V} = (A - BD^{-L}C) | \mathcal{V} \]

**Proof:** Since \(D\) has full column rank, it has a left inverse \(D^{-L}\). This also means that \(DG = 0\) implies \(G = 0\), and therefore the supremal controllability subspace

\[ \mathcal{R}^* = (A + BB^*P_{21}) \cap \{ Im(BG) \} = 0 \]

Therefore from (15) we get

\[ D^{-L} C | \mathcal{V} = -B^*P_{21} \]

and eventually

\[ (A + BB^*P_{21}) | \mathcal{V} = (A - BD^{-L}C) | \mathcal{V} \]

Similarly for \(\mathcal{V}\).

4 Controllability subspaces

**Nota bene:** throughout this section, when we write "\(P_X X\) is injective", we mean that the restricted projection operator \((P_X \mid | X)\) is injective.

**Definition 4.1** Let \(K_+\) and \(Q_+\) be right coprime and \(Q'Q'' = \left( \begin{array}{cc} Q_+ & 0 \\ 0 & R \end{array} \right)\) with \(R\) equivalent to a factor of \(Q_+\) (i.e. there exists a factorization \(Q_+ = Q_2 Q''\) where \(Q_2 = \left( \begin{array}{cc} Q' \ast & 0 \\ 0 & R \end{array} \right)\) and \(R\) is equivalent to \(Q'_2\)). Let \(Q'_2\) satisfy the equation \(KQ'' = \left( \begin{array}{cc} Q_+ & 0 \\ 0 & R \end{array} \right)\) left coprime, and define \(\overline{Q'_2} := \left( \begin{array}{cc} Q_+ & 0 \\ 0 & R \end{array} \right)\). We say that the factorization \(Q''\) is balanced with respect to \(K_+\) (and that the factorization \(Q''\) is balanced with respect to \(K_-\) if

\[ [Q'', R^e]_R = I \]

and

\[ [\overline{Q'_2}, R^e]_L = I \]

We also say that the seminvariant subspace \(H(K)Q''\) is the state space associated to the factorization.

**Theorem 4.1** Let \(Q'Q''\) be a balanced factorization w.r. to \(K_+\), and let \(\left( \begin{array}{cc} A & B \\ -B^* P^{-1} & I \end{array} \right)\) be a realization of \(K\); set \(X_R := P_{H(K)Q''} H(\overline{Q'_2})K_+\) and \(R := I_{A,B} X_R\). Then

\[ X_R = P_{H(K)Q''} H(Q')Q'' \cap P_{H(K)Q''} H(\overline{Q'_2})K_+ \]

and \(R\) is a controllability subspace for \(A, B\).
Sketch of the proof: we set \( X_V := P(H(K)Q^r)H(Q')Q'^r \) and \( X_H := P(H(K)Q^r)H(Q')K^e_+ \).

We know from Lemma 3.3 that \( V := I^1_{A,B}X_V \) is an antistabilizable controlled invariant subspace and that \( V := I^1_{A,B}X_T \) is a stabilizable controlled invariant subspace. Therefore the intersection is both stabilizable and antistabilizable; thus, if we show that \( V \cap T \) is controlled invariant we conclude that it is a controllable subspace. To do this, we show that \( X_R \subset X_V \cap X_T \), and that the projection \( P(H(K)Q^r)H(R^c) \) is injective; we then show that \( \dim R \geq \dim(V \cap \overline{V}) \). To show that \( R \) is controlled invariant, we show that \( P(H(K)Q^r)H(R^c) = P(H(K)Q^r)Q'^r P(H(K)H(Q'_2) \) for a suitable \( Q'_2 \); the conclusion then follows from Lemma 3.3. We refer to [6] for a detailed proof.

Proof: The only thing to check is that condition (17) is satisfied, so that we can apply the above theorem. But this follows from the fact that \( W_{-} := \begin{pmatrix} A & B \\ C & D \end{pmatrix} \) is full column-rank at infinity, and the output nulling condition \( [D_{-},0]G = 0 \) implies \([B_{-},0]G = 0 \).

References


