A PARAMETRIZATION OF EXTERNAL SPECTRAL FACTORS

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Abstract: We study the geometric structure of the spectral factors of a given spectral density Φ. We show that these factors can be associated to a set of invariant subspaces and we exhibit the manifold structure of this set, providing also an explicit parametrization for it, in the special case of coinciding algebraic and geometric multiplicity of the zeros of the maximum-phase spectral factor. We also make some connection with the set of solutions to the Riccati Inequality.

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1 Introduction

The characterization of all the minimum degree spectral factors W of a given p × p spectral density Φ of rank m₀ is a problem which has been widely studied in connection with the lattice of symmetric solutions to the Riccati Inequality. In particular, Anderson and Faurre gave a pre-studied in connection with the lattice of symmetrical factors are given by the solutions Φ of rank m which actually means that D in (1) is 0.

\[ \left[ \begin{array}{ccc} P - A P A' & G - A P C' \\ G' - C P A' & L - C P C' \end{array} \right] = \left[ \begin{array}{cc} B' & D' \end{array} \right] \]

for suitable matrices B, D which are determined up to a unitary transformation and can therefore always be chosen of full rank. Then the associated spectral factor is W(z) = D + C(z − A)^−1B (again up to a unitary transformation).

This characterization, though, does not provide much insight in the geometry of the set of spectral factors; moreover, it is not easy to characterize the elements on the boundary (that is the P for which rank([B', D']) drops). For the case of internal realizations (dim(W) = p × m₀) Lindquist and Picci have characterized the minimal stable spectral factors in terms of the coinner divisors of the minimum-phase spectral factor W₋. More recently the problem has been studied by Lindquist, Michalewicz and Picci [9] for external realizations.

We are not aware, though, of a geometric characterization of the set of all the spectral factors of Φ of a given dimension p × m. We exhibit here the manifold structure of this set, in the special case when algebraic and geometric multiplicity of the zeros of the maximum-phase spectral factor coincide, and we indicate how the corresponding set P of solutions to (1) can be obtained as a quotient modulo a group of (m − m₀) × (m − m₀) all-pass functions. For simplicity we only consider strictly proper factors, which actually means that D in (1) is 0.

2 Preliminaries and notation

We work with the row vectors Hardy spaces of the disk D. We define [8] L²ₚ(Π) to be the set of the square integrable m-dimensional row vector valued functions on the unit circle Π, and H²ₚ(Π²) to be the subspace of L²ₚ(Π) whose non negative (strictly negative) Fourier coefficients vanish. The functions in H²ₚ(Π²) are defined on Π, but they can be extended to analytic functions on the open complement E of the closure of the disk D (to the disk D), by taking the Cauchy integral:

\[ f(z) := \frac{1}{2\pi i} \int_{\Pi} \frac{f(e^{-i\omega})}{e^{i\omega} - z} \, d\omega \quad f \in H²ₚ \]

\[ g(z) := \frac{1}{2\pi i} \int_{\Pi} \frac{g(e^{i\omega})}{e^{i\omega} - z} \, d\omega \quad g \in \Pi² \]

The transposed of the function \((f^*)(z) := f(z)^{-1}\)T, where T denotes transposition, represents the extension in \(\Pi²\) of \(f(e^{i\omega})\). The inner product on \(H²ₚ\) thus becomes:

\[ (f, g) = \frac{1}{2\pi i} \int_{\Pi} g(z) f^*(z) \, dz \]
A \( p \times m \) matrix function \( F \) with rows in \( zH^2_m \) is said to be rigid if \( p \geq m \) and the product \( F^*(e^{i\omega})F(e^{i\omega}) = I_p \) a.e. (\( I_p \) is the identity) or if \( p \leq m \) and the product \( F(e^{i\omega})F^*(e^{i\omega}) = I_p \) a.e. It is inner (or stable all-pass) if \( p = m \).

The orthogonal projection onto a subspace \( X \) or onto the span of a vector \( b \) will be denoted by \( P^X \) and \( P^b \) respectively; in particular, projections onto \( H^2_m \) and \( \overline{H^2_m} \) will be denoted by \( P^{-} \) and \( P^{+} \) respectively. If \( B \) is an inner function in \( H^2_m \) with columns in \( zH^2_m \), we define \( H(B) := H^2_m \ominus \overline{H^2_m}B \). The shift on \( H^2_m \) is defined as \( Uf := z^{-1}f \). The adjoint is \( U^* = P^-M_z \), and clearly \( U^* \) is an isometric operator, i.e. \( U^*U = I \). A space \( X \in H^2_m \) is invariant, if \( UX \subset X \). Invariant subspaces are characterized by the Beurling Theorem (see e.g. [8]):

**Theorem 2.1** A subspace \( X \) is invariant for \( U \) in \( H^2_m \) if and only if \( X = H^2_mQ \) for some inner function \( Q \), which is uniquely determined up to a constant unitary factor.

The function \( Q \) becomes unique if a further condition is imposed, for example \( Q(e^{i\omega_0}) = T_0 \), where \( -\pi \leq \omega_0 < \pi \) is in the domain of analyticity of \( Q \) (or more generally if \( Q \) has radial limit in \( e^{i\omega} \)) and \( T_0 \) is unitary.

In our setting, since all the inner functions we will consider will be rational, it is not a restriction to make the following:

**Assumption 2.1** We will assume from now on that a coinvariant subspace \( Z = H(Q) \) is characterized by the unique inner function satisfying \( Q(1) = I \).

A subspace \( Y \) is said to be coinvariant, if \( U^*Y \subset Y \). We also say that \( X \subset Z \) is invariant in \( Z \) if \( P^ZUX \subset X \). Coinvariance in \( Z \) is defined analogously. A subspace \( X \subset H^2_m \) is called seminvariant if there exists a subspace \( Y \), such that \( Z := X \vee Y \) is coinvariant, and \( X \) is invariant in \( Z \). It can be shown (see [3]) that, although the subspace \( Y \) is not unique, there is only one such space which is orthogonal to \( X \), and it will be coinvariant. It is also easy to see this fact directly: define \( Z := \text{span}(\{U^nX; n \geq 0\}) \). If \( X \) is, according to the definition, invariant for \( U \) in \( Z \), its orthogonal complement in \( Z \) will be invariant for the adjoint \( U^* \) of \( U \), i.e. it is coinvariant. Therefore we have several corollaries to Beurling Theorem:

1. A space \( Y \) is coinvariant if and only if there exists an (essentially unique) inner function \( Q \) such that \( Y = H(Q) \).
2. A space \( Z \) is seminvariant if and only if there exist two (essentially unique) inner functions \( Q_1, Q_2 \) such that \( Z = H(Q_2)Q_1 \).

When these spaces are finite dimensional, the issue of how to represent the spaces \( H(Q) \) is addressed by the following well known result (see e.g. [5]):

**Theorem 2.2** Let \( X = H^2_m \ominus H^2_mQ \) be a coinvariant subspace in \( H^2_m \); then there exists a controllable pair \( A, B \) such that the rows of \( (z-A)^{-1}B \) form a basis \( x_1, x_2, ..., x_n \) in \( X \); the inner product matrix \( P \) defined as \( p_{ij} := (x_i, x_j) \) satisfies the equation:

\[
P = AP^+ + BB^*\]

Moreover, \( Q(z) = D + C(z-A)^{-1}B \) where \( C, D \) are the unique solution to

\[
AP^* + BD^* = 0
\]

\[
CPC^* + DD^* = I
\]

\[
Q(1) = D + C(I-A)^{-1}B
\]

Remark: the usual formulation is for an observable pair \( A, C \) and the Lyapunov equation has \( A^* \) on the other side. The choice of the point 1 for the normalization condition (4) is quite arbitrary.

We define now minimum-phase functions. An element \( f \in H^2_m \) is generating an invariant subspace \( X \), if \( X \) is the minimal invariant subspace containing \( f \). An invariant subspace in \( H^2_m \) has multiplicity \( m_0 \) if it can be generated by no less than \( m_0 \) vectors. It is maximal if there is no subspace \( Y \supset X \) of the same multiplicity \( m_0 \). A matrix function is minimum-phase, or outer in \( H^2_m \) if its rows generate a maximal subspace.

Let \( \Phi \) be a rational spectral density, i.e. a symmetric \( p \times p \) rational function of rank \( m_0 \) which is nonnegative definite on the circle \( T \). It is well known that for any \( m > m_0 \) there are infinitely many factorizations \( \Phi = W \Phi^* + W \Phi^* \), but there exists an essentially unique factorization \( \Phi = W_-W_+^* \), with \( W_- \) minimum-phase in \( H^2_{m_0} \). Similarly there is a unique factorization (still up to a unitary transformation) \( \Phi = W_+W_+^* \) where \( W_+ \in T^2_{m_0} \) with \( W_+ \) conjugate minimum-phase. Both this factorizations have dimension \( p \times m_0 \). Moreover, it can be shown (see [7]) that there exists an essentially unique inner function \( K_+ \) of dimension \( m_0 \times m_0 \) of minimal degree such that the rows of

\[
W_+ := W_+ K_+ \tag{5}
\]

are in \( H^2_{m_0} \); the matrix function \( W_+ \) is called the maximum-phase spectral factor. The factorization (5) is also known as Douglas-Shapiro-Shield factorization (in this case of \( W_+ \) in \( H^2_{m_0} \)).

The unitary transormation up to which the spectral factor is determined, can be fixed by selecting a particular basis in \( H^2_m \) for each \( m \); the basis we choose is such that \( H^2_{m_0} \) and \( H^2_{m_0-m_0} \) are imbedded in \( H^2_m \) by the following maps:

\[
H^2_{m_0} \mapsto [H^2_{m_0}, 0] \subset H^2_m
\]

\[
H^2_{m_0-m_0} \mapsto [0, H^2_{m_0-m_0}] \subset H^2_m
\]

We still have to fix a basis in the two subspaces \( H^2_{m_0} \) and \( H^2_{m_0-m_0} \). We can assume that in \( H^2_m \) a
particular factorization $W_+$ of $\Phi$ is given; we will see shortly that we can impose conditions so that all the other spectral factors can be chosen uniquely.

We recall that a scalar zero of a rational transfer function $W$ is a complex number $z$ such that the matrix
\[
\begin{bmatrix}
\zeta f - A & B \\
-C & D
\end{bmatrix}
\]
associated to any minimal realization of $W$ has a rank drop in $z$. It is well known that if $z_1$ is unstable, this implies that there exists a $b_1 \in H^2_m$ such $Wb_1 \frac{1}{z - z_1}$ is analytic in the complement of the closed unit disk.

In the Hilbert space setting, this is to say that $z_1 = b_1 \frac{1}{z - z_1}$ is orthogonal to $W$ in $H^2_m$. In particular, $z_i = b_i \frac{1}{z - z_i}$ is an unstable zero of $W_+$ if $b_i$ is a row vector with components $\Phi^{-m_0} z_i$ is a scalar zero of $W_+$ and the rows of $W_+$ are orthogonal to $z_i$.

This is not a new definition of zero; it is simply an adaptation to the Hilbert space setting of the standard definitions existing in the literature (see e.g. [12]). In particular, the set of unstable zeros generates the equivalent of the zero module for $W_+$.

Going back to a general $W$, we can try to extend a definition of unstable zero along the above lines. The relation between minimal spectral factors has been characterized by Lindquist and Picci.

**Theorem 2.3** Let $W$ be a spectral factor of minimal McMillan degree. Then there exist (essentially unique) rigid functions $\hat{Q}_W$ and $\hat{\bar{Q}}_W$, of dimension $m \times m$, such that $W\hat{Q}_W = [W_+, 0]$ and $W\hat{\bar{Q}}_W^* = [W_-, 0]$. Moreover,
\[
Q_W\hat{Q}_W = \begin{bmatrix} Q_+ & \hat{Q}_- \end{bmatrix}
\]
where $Q_+$ is independent of the choice of $W$, $\hat{Z} = \mathbf{P}^{b_{m_0} H(Q_W)}$ and $\hat{\bar{Q}}_-$ is the inner function associated to $\hat{Z}$.

The function $Q_+$ is also relating the minimum-phase and maximum-phase factors, by the well known relation
\[
W_+ = W_- Q_+
\]
The function $Q_+$ will play an important role in the following.

**3 Main Results**

As announced, we will assume for simplicity that $W$ is strictly proper, and that $zW$ is regular at infinity. The results are much simpler to state, and it is not difficult to extend them to the general case, as will be done in [2]. Therefore, in this section, by state space for a given spectral factor $W$ we mean a semiinvariant subspace $X$ in $H^2_m$ such that the rows of $W$ belong to $X$. It can be shown that in general (in the external case) for each spectral factor there exists a unique minimal state space (in the sense that it has dimension as small as possible: it can be shown [11] that this dimension is always $n$). By minimal spectral factor we therefore mean a factor of McMillan degree $n$. Denote by $X_+$ the state space of the maximum-phase spectral factor, i.e.
\[
x_+ = \text{span}(\mathbf{P}^{-U^n} W_+)
\]
Let $Z$ be coinvariant in $H^2_m$ and $Z \perp W_+$, where with the abuse of notation we intend that $Z$ is orthogonal to the rows of $W_+$. We set $X_Z := (X_+ \lor Z) \cap Z$.

**Lemma 3.1** $X_Z$ is a semiinvariant minimal subspace containing $W_+$.

**Proof:** $X_+ \lor Z$ is coinvariant in $H^2_m$, since both $X_+$ and $Z$ are. Since $Z$ is coinvariant, it is also coinvariant in $X_+ \lor Z$. Thus its orthogonal complement $X_Z$ is semiinvariant in $X_+ \lor Z$. Since $Z \perp W_+$,
$W_+ \in X_+$, and in view of the dimension, $X_+$ is minimal.

Denote now by $Z_+$ the maximal coinvain subspace in $H^2_{m_0}$ orthogonal to $W_+$. Since we know from Corollary 2.1 that there exists a unique inner function $Q_+$ such that $W_+ = W_+ Q_+$, it is clear that $Z_+ = H(Q_+)$.

We need to clarify the link between $W$ and the space $Z$. We will assume, for the time being, that $Z$ is a minimal set of zeros, i.e. $\dim Z = \dim P^H_{m_0} Z$.

**Theorem 3.1** There is a one to one correspondence between minimal spectral factors $W$ and minimal coinvain subspaces $Z$ such that $P^H_{m_0} Z \subset Z_+$.

**Proof:** given $W = [W_+ , 0](Q_W)^\ast$, we have set by definition $Z_W = H(Q_W)$. Conversely, given $Z$ such that $P^H_{m_0} Z \subset Z_+$, we know from Corollary 2.1 that $H(Q)$ and $Q(1) = I$. Moreover, the elements of $H(Q)$ are orthogonal to the rows of $[W_+ , 0]$, and therefore $Q$ divides $[W_+ , 0]$ on the right. The desired factor is then given by $W = [W_+ , 0](Q_W)^\ast$.

How do we characterize all the $W$? From the above theorem we need to characterize all the $Z$ such that $P^H_{m_0} Z \subset Z_+$ and is invariant in $Z_+$.

**Lemma 3.2** Let $Z = H(Q)$ be a coinvain subspace in $H^2_m$ of dimension $r$ such that $P^H_{m_0} Z \subset Z_+$, and let $A_{z_1}$, $B_{z_1}$, $C_{z_1}$, $D_{z_1}$ be a controllable realization for $Q_+$; then there exists a unique matrix $\tilde{B}$ such that $Z := (z - A)^{-1}[B_{z_1}, \tilde{B}]$ is a basis for $Z$. Conversely, any matrix $\tilde{B} \in \mathbb{R}^{r \times (m - m_0)}$ yields a space satisfying the above condition.

**Proof:** let $A_Q$, $B_Q$, $C_Q$, $D_Q$ be a minimal realization of $Q$. From Lemma 2.2 we know that a basis for $H(Q)$ is $(z - A)^{-1}B$; it is also obvious that we can partition $B_Q = [B_Q, \tilde{B}_Q]$ where the dimensions are $m_0$ and $m - m_0$, and that

$$Z_+ = P^H_{m_0} H(Q) = \text{span} \{([z - A_Q])^{-1} \tilde{B}_Q] ; i = 1, \ldots, r\}$$

so, $(A_Q, \tilde{B}_Q$ and $(A_{z_1}, B_{z_1})$ are related by a similarity transformation, which is unique in view of controllability. Therefore, $\tilde{B}$ is also uniquely determined. The converse is also trivially following from controllability of $(A_{z_1}, B_{z_1})$.

We denote by $\mathcal{H}_m$ the set of linear spaces with rows in $H^2_m$, endowed with the gap topology:

$$d(M, N) := \|P^M - P^N\| \quad M, N \in \mathcal{H}_m \quad (7)$$

and set, for a given basis $z_1 = (z - A_{z_1})^{-1} B_{z_1}$ of $Z_+$ corresponding to an arbitrary realization $(A_{z_1}, B_{z_1})$ of $Q_+$,

$${\tilde{B}}_{z_1} := \{B \in \mathbb{C}^m ; B = [B_{z_1}, \tilde{B}]\}$$

and

$$\tilde{Z}_m := \{Z \in \mathcal{H}_m : P^H_{m_0} Z = Z_+\}$$

**Theorem 3.2** The space $\tilde{Z}_m$ is homeomorphic to $\mathbb{C}^{r \times (m-m_0)}$.

**Proof:** from the above lemma, for a given choice of basis in $Z_+$, we can associate to each element $Z \in \tilde{Z}_m$ a unique matrix $B$ such that $[B_{z_1}, \tilde{B}] \in \tilde{B}_{z_1}$. From [1], the map from $((A_Q, B_Q)$ controllable; $B_Q \in \mathbb{C}^{r \times m_0})$ to the set of $T^m$ inner functions of degree $r$, endowed with the $L^2$ norm on the rows, is smooth, and therefore so is its restriction to the open subset (in the induced topology) $\{(A, B_Q) \in \tilde{B}_m \}$ to the image. Therefore, our task reduces to prove that the map $\phi$ from the image of $\tilde{B}_{z_1}$ to $\tilde{Z}_m$ defined as $\phi(Q) = H(Q)$ is continuous.

By construction the map is bijective. Let first $(Q_n - Q) \to 0$; then, for $x \in H^2_n$, $\|xH(Q_n) - P^H(Q_n)\|$ is bounded by the quantity $\|x(Q_n^{-1} - Q^{-1})P^H_{m_0}(Q_n - Q)\|$ which converges to zero as $(Q_n - Q)$ does.

Conversely, let $\|P^H(Q_n) - P^H(Q)\| \to 0$; we want show that in the limit also $\|Q_n - Q\|$ is vanishing. Let $x \in H^2_n$ Then

$$\|xQ[P^H(Q_n) - P^H(Q)]\| =\|xQ((Q_n^{-1} - Q^{-1})P^H_{m_0}(Q_n - Q))\| + \|xQ^{-1}P^H_{m_0}(Q_n - Q)\|$$

that is, the operator $\|QQ_n^{-1}P^H_{m_0}\|$ converges to zero strongly as $\|P^H(Q_n) - P^H(Q)\| \to 0$. Since this is a Hankel operator, this means that the strictly antistable part of $QQ_n^{-1}$ vanishes. In the same manner we can conclude that $\|Q_nQ_n^{-1}P^H_{m_0}\|$ is strongly convergent to zero, and therefore the strictly stable part of $QQ_n^{-1}$ is also vanishing. But this means that $QQ_n^{-1}$ converges to a constant, and since the functions are inner and take the value $I$ in $1$, this constant is necessarily the identity, as wanted.

The structure of $\tilde{Z}_m$ is quite complicated to describe; nevertheless this is not so crucial, since we are actually interested in equivalence classes of this set, which have a better behaviour.

We define the following equivalence relation $\sim$ on $\tilde{Z}_m$:

$$Z_1 \sim Z_2 \text{ if both}$$

$$Z_1 \in \{Z_1 \cap H^2_{m-m_0} = Z_2 \cap (Z_2 \cap H^2_{m-m_0}) \} \quad (8)$$

The interest of this relation is explained in the following simple results.

**Lemma 3.3** Let $Z_1, Z_2 \in \tilde{Z}_m$, and let $Q_1, Q_2$ the associated inner matrices. Then $Z_1$ is equivalent to $Z_2$ if and only if there exists an $(m - m_0) \times (m - m_0)$ all-pass matrix $Q$ such that $Q_1 = Q_2 \left[ \begin{array}{cc} I_{m_0} & 0 \\ 0 & Q \end{array} \right]$.
Proof: by construction, \( Z_1 \cap H^2_{m-m_0} \) is coinvariant, and it has associated an inner function of the form

\[
\begin{bmatrix}
I_{m_0} & 0 \\
0 & Q_1^I
\end{bmatrix},
\]

and \( Q_1 \) factors as \( Q_1' \begin{bmatrix}
I_{m_0} & 0 \\
0 & Q_1^I
\end{bmatrix}. \)

Similarly, \( Q_2 \) factors as \( Q_2' \begin{bmatrix}
I_{m_0} & 0 \\
0 & Q_2^I
\end{bmatrix}. \) Then the relation (8) in terms of inner functions writes

\[
H(Q_1') \begin{bmatrix}
I_{m_0} & 0 \\
0 & Q_1^I
\end{bmatrix} = H(Q_2') \begin{bmatrix}
I_{m_0} & 0 \\
0 & Q_2^I
\end{bmatrix}
\]

which implies the conclusion.

Notice that all the matrices we are considering are normalized, i.e. \( Q(1) = I \).

Clearly for each equivalence class of \( \tilde{Z}_m \) there exists a unique element \( Z \) such that \( Z \cap H^2_{m-m_0} = 0 \); so \( \dim \tilde{Z} = \dim \mathbb{P} H^2_{m_0} \). We take this element as the representative of the equivalence class in \( \tilde{Z}_m \).

Corollary 3.1 Let \( Z_1 = H(Q_1) \) and \( Z_2 = H(Q_2) \).

Then \( Z_1 \sim Z_2 \) if and only if \( Q_1 = Q_2 \).

We now define \( \tilde{Z}_m := \tilde{Z}_m/\sim \).

Corollary 3.2 let \( \tilde{Q}_m \) be the set of \( m \times m_0 \) rigid function \( Q \) such that \( W = W_+ Q^* \) has rows in \( H^2 \), and \( W(1) = W_+ (1) \). Then the completion \( \tilde{Q} \) for which \( Q = [\tilde{Q}, \bar{Q}] \) is minimal inner is such that \( Z = H(Q) \in \tilde{Z}_m \) and the completion map from \( \tilde{Q}_m \) to \( \tilde{Z}_m \) is continuous.

Proof: note first that \( Q \) and \( \tilde{Q} \) have the same McMillan degree. In fact, if \( (A, [\tilde{B}, \bar{B}]) \) is a realization of \( Q \) such that \( (A, \tilde{B}, \bar{B}) \) is not controllable, we can find a change of basis for which \( A \) is lower triangular Jordan, and \( \bar{B} \) has a zero row in correspondence of the uncontrollable mode \( a_i \); if \( \bar{b}_i \) is the corresponding row of \( \bar{B} \), we know that \( H(Q) \ni (z-a_i)^{-1} \bar{b}_i \), and therefore, defining \( B_0(z) := P \bar{b}_i z^{-a_i} \) it is \( Q B_0(z) \in H^2 \), contradicting minimality of the extension.

Therefore, if \( \{ \tilde{Q}_n \} \) is a Cauchy sequence converging to \( \bar{Q} \), the completion \( \bar{Q} \) has always degree less than or equal to the limit of the degrees of the sequence. If it is equal, there is nothing to show: the topology coincides with that of inner functions. If there is a degree drop, it means that the limit \( Q \) of \( \{ Q_n \} \) in the usual topology can be factored as

\[
\begin{bmatrix}
I & 0 \\
0 & Q_1
\end{bmatrix},
\]

and therefore \( Q \) is the representative of the equivalence class of \( \bar{Q} \).

Theorem 3.3 Let \( A_{z_0} \) be diagonalizable. Then the set \( \tilde{Z}_m \) is a manifold diffeomorphic to the product \( S^r_{m-m_0} \) of \( r (m-n) \) - dimensional spheres.

Proof: we prove continuity, and refer to [2] for smoothness, since the construction of the charts is quite complex. Let \( (A_{z_0}, B_{z_0}) \) be a realization of \( Q_+ \) such that \( A_{z_0} \) is diagonal and the rows of \( B_{z_0} \) have norm 1; we represent \( S^r_{m-m_0} \) as \( (\Delta, \tilde{B}) \) where

\[
\Delta = \text{diag}\{\delta_1, ..., \delta_r\} \quad \text{with } \delta_i \text{ in the one point compactification of the unit disk } S_1 := \{z; |z| < 1 \cup \{1\} \}
\]

(we are working over the complex field) and \( b_i \in S_{m-m_0-1} \) and we consider the following map \( \phi \) from \( S^r_{m-m_0} \) to \( \tilde{Z}_m \):

\[
\phi(\Delta, \tilde{B}) := \text{span}\{(z - A_{z_0})^{-1} [\Delta^2 B_{z_0} + \Delta (I - (\Delta^*)^{1/2}) \tilde{B}]\}
\]

The map is clearly continuous, since \( (A_{z_0}, B_{z_0}) \) can be completed smoothly. It is bijective: since for any \( Z \in \tilde{Z}_m \) of dimension \( s \leq r \), \( \dim \mathbb{P} H^2_{m_0} Z = s \), \( Z \) is generated by eigenvectors of \( U_{m_0}^* Z \), which are of the form \( (z - a_i)^{-1} \tilde{b}_i \), with \( b_i = 0 \) if \( \beta_i = 0 \). Now comparing with (9), it is seen that

\[
|\delta_i| = \frac{\|\tilde{b}_i\|}{\|\tilde{b}_i\| + \|\tilde{b}_i\|}
\]

and that for \( 0 < |\delta| < 1 \)

\[
\arg \delta_i = \frac{\arg \tilde{b}_i}{2 \arg \tilde{b}_i}
\]

where \( \tilde{b}_i \) is a nonzero component of \( \tilde{b}_i \); then \( \tilde{b}_i \) is determined, for \( 0 < |\delta| < 1 \).

\[
\delta_i (1 - |\delta_i|) = \tilde{b}_i
\]

and is zero otherwise. So the map \( \phi \) is onto, and since the solution is obviously unique, it is also one to one. Continuity of \( \phi^{-1} \) is obvious for \( |\delta| < 1 \), i.e. when \( \|\tilde{b}_i\| \) remains bounded. In the case \( \|\tilde{b}_i\| \to \infty \), the basis for \( Z \) contains the vector \( (z - a_i)^{-1} \tilde{b}_i \), so that we can substitute zeros in the \( i \)-th row for \( \beta_i \). But the element \( \delta_i \) in the inverse image is also converging to zero, and so the map \( \phi \) is a homeomorphism.

We now come to the main result of the paper, which is to represent the set \( P \) as the quotient of a smooth manifold. In our simplified setting, the set of solutions \( P \) of (1) is restricted those for which \( D = 0 \); each \( P \in \mathcal{P} \) determines a set of solutions \( B_P \) of (1) and an equivalence class \( W_P \) of spectral factors \( W = C(z - A)^{-1} B, B \in B_P \) where the equivalence relation is multiplication by a unitary matrix on the right. Define

\[
\tilde{Z}_m := \tilde{Z}_m / G_m
\]

where we have set

\[
G_m := \begin{bmatrix}
I & 0 \\
0 & U(m-m_0)
\end{bmatrix}
\]

and set

\[
\mathcal{P}_m := \{ P \in \mathcal{P} : \text{rank}(P - APA^*) \leq m \}
\]

Theorem 3.4 For each \( m \) there is a homeomorphism

\[
\pi_m : \mathcal{P}_m \mapsto \tilde{Z}_m
\]
such that

$$\pi_m : P_m \mapsto Z_m$$

is a diffeomorphism

i) the points such that $\text{rank}(P - APA^*) = s < m$ correspond to the points of $Z_m$ whose centralizer has dimension $m - s$

ii) the extremal points for which $\text{rank}(P - APA^*) = m_0$ (internal realizations) correspond to fixed points of $G_m$

**Proof:** (sketch: see [2] for details) to each $P \in \mathcal{P}$ there corresponds an equivalence class $W_P$ of spectral factors. It has been shown that $Z_m$ is the manifold of inner functions satisfying the condition $Q(1) = I$ and yielding the factorization $WQ = [W_+, 0]$; let $T \in G_m$; then also $WTT^*QT = [W_+, 0]$ and satisfies the normalizing condition; therefore all the conjugates of $Q$ will yield $W$ in the same equivalence class. These are the only ones, since $[W_+, 0]T = [W_+, 0]$ and $T$ unitary forces $T \in G_m$. In conclusion, $\pi_m(P)$ associates to an element $P$ the set of spectral factors in $W_P \cap Z_m$, and these sets are disjoint orbits of $G$. This shows that $\pi$ is a map. Invertibility follows from direct computation: if $Z \in Z_m$, then $Z = H(Q)$ and $WQ^*$ is a stable spectral factor admitting a minimal realization $W = zC(z - A)^1B$ with $(A, C)$ independent of $W$, and $P$ is then the controllability gramian of $(A, B)$. It is continuous, since the set of solutions to the positive real equations depend bicontinuously on the coefficients. To see that it is a diffeomorphism, state space formulas are needed and we refer to [2].

About i) and ii), it is clear that if $W = C(z - A)B$, with $\text{rank} \ B = s$, then its centralizer has dimension $m - s$; in particular, for internal realizations, this dimension is $m - m_0$ and therefore the centralizer is the whole $G_m$.

Notice that, in particular, the the lattice of symmetric solutions to (1) $\mathcal{P} = \mathcal{P}_{n+p}$ is homeomorphic to $Z_{m+p}$.

**References**


