On the parametrization of Schur functions of degree \( n - 1 \) with \( n \) fixed interpolating conditions

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Abstract

We investigate here a parametrization of Schur functions of degree \( n - 1 \) with given \( n \) interpolating conditions. The results are presented for scalar case.

1 Introduction

We investigate here the problem of parametrizing the set of Schur function of degree \( n - 1 \) which satisfies a set of \( n \) given interpolating conditions. The results are based on a refinement of [5] where the general degree \( n \) case is treated. General results for the degree \( n \) case are due to Kimura and Georgiou (see [4], [3]) and more recently to Byrnes, Georgiou and Lindquist (see [2] and reference therein). We derive here, similarly to [5], a parametrization for scalar Schur functions of degree \( n - 1 \) which can be extended to positive real functions.

2 Preliminaries and notation

We shall denote by \( \mathbb{C}^+ \) the right half-plane, and by \( \mathcal{H}^2_+ \) the corresponding Hardy space of vector or matrix valued functions (the proper dimension will be understood from the context). Let \( F,G \) be \( p \times m \) matrix valued functions in \( \mathcal{H}^2_+ \). The space \( \mathcal{H}^2_+ \) is naturally endowed with the scalar product,

\[
<F,G> = \frac{1}{2\pi} \text{Tr} \int_{-\infty}^{\infty} F(iy)G(iy)^* \, dy,
\]

and we shall denote by \( || \cdot ||_2 \) the associated norm. Note that if \( M \) is a complex matrix, \( \text{Tr} \) stands for its trace, \( M^T \) for its transpose and \( M^* \) for its transpose conjugate. Similarly, we define \( \mathcal{H}^\infty_+ \) to be Hardy space of essentially bounded functions analytic on the right half plane. A \( p \times m \) matrix valued function \( Q \in \mathcal{H}^\infty_+ \) is called a Schur function if \( Q(\omega)Q(\omega)^* \leq I_p \) for (almost all) \( \omega \in \mathbb{R} \); it is called wide inner if \( p < m \) and \( Q(\omega)^*Q(\omega) = I_p \) for (almost all) \( \omega \in \mathbb{R} \). The set of Schur functions of degree \( n \) and dimension \( p \times m \) with constant term \( D \) is denoted by \( \mathcal{S}^{n,m}_+(D) \). Similarly, \( \mathcal{Q}^{n,m}_+(D) \) will denote the subset of functions in \( \mathcal{S}^{n,m}_+(D) \) which are inner.

We assume that we are given a set of interpolation points \( s_1,\ldots,s_n \) in \( \mathbb{C}^+ \) and interpolating conditions

\[
U = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}, \quad V = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}
\]

with \( u_i \), \( v_i \) row vectors in \( \mathbb{C}^m \) and \( \mathbb{C}^p \), respectively, \( \|u_i\| = 1 \) and \( \|v_i\| \leq 1 \) for \( i = 1,\ldots,n \) and we want to find the solutions \( Q \) to the problem

\[
u_i Q(s_i)^* = v_i, \quad i = 1,\ldots,n
\]

which are Schur functions of degree \( n - 1 \).

We will consider the following more general problem:

Problem 1 Given a \( \mathcal{A} \) stable and interpolating directions \( u_1,\ldots,u_n \) and conditions \( v_1,\ldots,v_n \) and a constant matrix \( D \) parametrize all functions \( Q \) of degree \( n - 1 \) satisfying

\[
\int_{\Gamma} (Q(s)U^* - V^*)(sI + \mathcal{A}^*)^{-1} ds = 0
\]

where \( \Gamma \) is a closed curve around \( \sigma(-\mathcal{A}^*) \).

The set of Schur functions of degree \( n - 1 \) satisfying (4), endowed with the induced \( \mathcal{H}^\infty_+ \) topology will be denoted by \( \mathcal{S}_{n-1}(\mathcal{A},U,V,D) \).

3 Main results

Given some data \( \mathcal{A},U,W \) we define the Pick matrix \( \mathcal{R} \) to be the solution to

\[
\mathcal{A} \mathcal{P} + \mathcal{R} \mathcal{A}^* + UU^* - WW^* = 0
\]

We start with the following simple
Lemma 3.1 Let $A, U, W, D$ be given and suppose $Q = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. Then $Q$ is an interpolating function, i.e., it satisfies (4) if and only if there exists $Y$:

\[
\begin{aligned}
AY + YA^* + BU^* &= 0 \\
CY &= DU^* - W^*
\end{aligned}
\]

Lemma 3.2 Let $\mathcal{A}, U, V$ and $\tilde{V}_0^T = [\tilde{v}_1, \tilde{v}_2, ..., \tilde{v}_{n-1}, 0]^T$ be given such that, setting $W = [V, \tilde{V}_0]$, the equation (7) has a positive definite solution. Then there are at most two values $\tilde{v}_n$ such that, setting $\tilde{V}^T = [\tilde{v}_1, \tilde{v}_2, ..., \tilde{v}_{n-1}, \tilde{v}_n]^T$ and $W = [V, \tilde{V}]$, the solution $\mathcal{R}$ to (5) is singular. If all the coefficients are real, then also these values are real.

The above is a parametrization of all inner completions of Schur interpolants; since each interpolants has many completions, corresponding to the solutions to BRL equations, this result does not yield a parametrization of $S_n(\mathcal{A}, U, V, D)$; the parametrization is simply achieved by choosing one such completion, e.g. the maximum-phase one. The result is quite straightforward and is omitted for lack of space.

References


