Model reduction of large-scale systems
An overview and some new results

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Outline

1. Overview
2. Motivation
3. Approximation methods
   - SVD-based methods
   - Krylov-based methods
4. Choice of projection points
   - Passivity preserving model reduction
   - Optimal $\mathcal{H}_2$ model reduction
5. Model reduction from measurements
   - S-parameters
   - Classical realization theory
   - Finite data points: the Loewner matrix
   - Tangential interpolation: $\mathbb{L}$ & $\sigma \mathbb{L}$
   - Recursive Loewner-matrix framework
6. Conclusions
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The big picture

- Physical System
- Modeling
- ODEs
- discretization
- PDEs
- Model reduction
- Data
- and/or
- reduced # of ODEs
- Simulation
- Control

Model reduction of large-scale systems
Dynamical systems

We consider explicit state equations

\[ \Sigma : \quad \dot{x}(t) = f(x(t), u(t)), \quad y(t) = h(x(t), u(t)) \]

with state \( x(\cdot) \) of dimension \( n \gg m, p \).
**Problem statement**

**Given**: dynamical system

\[
\Sigma = (f, h) \text{ with: } u(t) \in \mathbb{R}^m, \ x(t) \in \mathbb{R}^n, \ y(t) \in \mathbb{R}^p.
\]

**Problem**: Approximate \(\Sigma\) with:

\[
\hat{\Sigma} = (\hat{f}, \hat{h}) \text{ with: } u(t) \in \mathbb{R}^m, \ \hat{x}(t) \in \mathbb{R}^k, \ \hat{y}(t) \in \mathbb{R}^p, \ k \ll n:\]

(1) Approximation error small - global error bound
(2) Preservation of stability/passivity
(3) Procedure must be computationally efficient
Approximation by projection

**Unifying feature** of approximation methods: projections.

Let $V, W \in \mathbb{R}^{n \times k}$, such that $W^* V = I_k \Rightarrow \Pi = VW^*$ is a projection. Define $\hat{x} = W^* x$. Then

$$\dot{\hat{x}}(t) = W^* f(V\hat{x}(t), u(t))$$
$$y(t) = h(V\hat{x}(t), u(t))$$

Thus $\hat{\Sigma}$ is "good" approximation of $\Sigma$, if $x - \Pi x$ is "small".
Special case: linear dynamical systems

\[ \Sigma: \dot{E}x(t) = Ax(t) + Bu(t), \ y(t) = Cx(t) + Du(t) \]

\[ \Sigma = \begin{pmatrix} E & A \\ C & D \end{pmatrix} \]

**Problem**: Approximate \( \Sigma \) by projection: \( \Pi = VW^* = \)

\[ \hat{\Sigma} = \begin{pmatrix} \hat{E}, \hat{A} \\ \hat{C} & \hat{D} \end{pmatrix} = \begin{pmatrix} W^*EV, W^*AV \\ CV & D \end{pmatrix}, \ k \ll n \]

**Norms**:
- \( H_\infty \)-norm: worst output error \( \|y(t) - \hat{y}(t)\| \) for \( \|u(t)\| = 1 \).
- \( H_2 \)-norm: \( \|h(t) - \hat{h}(t)\| \)
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## Motivating Examples: Simulation/Control

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<th>Motivating Example</th>
<th>Examples</th>
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</table>
| **1. Passive devices** | - VLSI circuits  
- Thermal issues  
- Power delivery networks |
| **2. Data assimilation** | - North sea forecast  
- Air quality forecast |
| **3. Molecular systems** | - MD simulations  
- Heat capacity |
| **4. CVD reactor** | - Bifurcations |
| **5. Mechanical systems:** | - Windscreen vibrations  
- Buildings |
<p>| <strong>6. Optimal cooling</strong> | - Steel profile |
| <strong>7. MEMS: Micro Electro-Mechanical Systems</strong> | - Elf sensor |
| <strong>8. Nano-Electronics</strong> | - Plasmonics |</p>
<table>
<thead>
<tr>
<th>Time Period</th>
<th>Description</th>
<th>Details</th>
</tr>
</thead>
</table>
| 1960’s: IC        | 1971: Intel 4004                          | 10\(\mu\) details  
2300 components
64KHz speed       |
|                   | 2001: Intel Pentium IV                    | 0.18\(\mu\) details  
42M components
2GHz speed  
2km interconnect
7 layers          |
Motivation

Passive devices: VLSI circuits

<table>
<thead>
<tr>
<th>Technology (μm)</th>
<th>Delay (ns)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>1.0</td>
</tr>
<tr>
<td>0.08</td>
<td>0.8</td>
</tr>
<tr>
<td>0.1</td>
<td>1.0</td>
</tr>
<tr>
<td>0.08</td>
<td>0.8</td>
</tr>
</tbody>
</table>

Average Wiring Delay ≈ 0.25 μm

Today’s Technology: 65 nm

65nm technology: gate delay < interconnect delay!

Conclusion: Simulations are required to verify that internal electromagnetic fields do not significantly delay or distort circuit signals. Therefore interconnections must be modeled.

⇒ Electromagnetic modeling of packages and interconnects ⇒ resulting models very complex: using PEEC methods (discretization of Maxwell’s equations): $n \approx 10^5 \ldots 10^6$ ⇒ SPICE: inadequate
# Mechanical Systems: Buildings

## Earthquake prevention

<table>
<thead>
<tr>
<th>Building</th>
<th>Height</th>
<th>Control mechanism</th>
<th>Damping frequency</th>
<th>Damping mass</th>
</tr>
</thead>
<tbody>
<tr>
<td>CN Tower, Toronto</td>
<td>533 m</td>
<td>Passive tuned mass damper</td>
<td>0.14 Hz, 2x300t</td>
<td>200t</td>
</tr>
<tr>
<td>Hancock building, Boston</td>
<td>244 m</td>
<td>Two passive tuned dampers</td>
<td>0.10 Hz, 2x100t</td>
<td>100t</td>
</tr>
<tr>
<td>Sydney tower</td>
<td>305 m</td>
<td>Passive tuned pendulum</td>
<td>0.33-0.62 Hz, 270t</td>
<td>270t</td>
</tr>
<tr>
<td>Rokko Island P&amp;G, Kobe</td>
<td>117 m</td>
<td>Passive tuned pendulum</td>
<td>0.185 Hz, 340t</td>
<td>180t</td>
</tr>
<tr>
<td>Yokohama Landmark tower</td>
<td>296 m</td>
<td>Active tuned mass dampers (2)</td>
<td>0.3 Hz, 300t</td>
<td>300t</td>
</tr>
<tr>
<td>Shinjuku Park Tower</td>
<td>296 m</td>
<td>Active tuned mass dampers (3)</td>
<td>0.18 Hz, 340t</td>
<td>180t</td>
</tr>
<tr>
<td>TYG Building, Atsugi</td>
<td>159 m</td>
<td>Tuned liquid dampers (720)</td>
<td>0.3 Hz, 300t</td>
<td>300t</td>
</tr>
</tbody>
</table>

- **Taipei 101**: 508m
- Damper between 87-91 floors
- 730 ton damper
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Approximation methods: Overview

Krylov
- Realization
- Interpolation
- Lanczos
- Arnoldi

SVD

Nonlinear systems
- POD methods
- Empirical Gramians

Linear systems
- Balanced truncation
- Hankel approximation

Krylov/SVD Methods
SVD Approximation methods

A prototype approximation problem – the SVD
(Singular Value Decomposition): \( A = U \Sigma V^* \).

Singular values provide trade-off between accuracy and complexity
POD: Proper Orthogonal Decomposition

Consider: $\dot{x}(t) = f(x(t), u(t)), \ y(t) = h(x(t), u(t))$.

**Snapshots of the state:**

$$\mathcal{X} = [x(t_1) \ x(t_2) \ \cdots \ x(t_N)] \in \mathbb{R}^{n \times N}$$

**SVD:** $\mathcal{X} = UV^* \approx U_kV_k^*, \ k \ll n$. Approximate the state:

$$\hat{x}(t) = U_k^*x(t) \ \Rightarrow \ x(t) \approx U_k\hat{x}(t), \ \hat{x}(t) \in \mathbb{R}^k$$

Project state and output equations. Reduced order system:

$$\dot{\hat{x}}(t) = U_k^*f(U_k\hat{x}(t), u(t)), \ y(t) = h(U_k\hat{x}(t), u(t))$$

$\Rightarrow \hat{x}(t)$ evolves in a low-dimensional space.

**Issues with POD:**

(a) Choice of snapshots, (b) singular values not I/O invariants.
SVD methods: the Hankel singular values

Trade-off between accuracy and complexity for linear dynamical systems is provided by the **Hankel Singular Values**, which are computed (for stable systems) as follows:

Define the **gramians**

\[
P = \int_0^\infty e^{At}BB^*e^{A^*t} \, dt, \quad Q = \int_0^\infty e^{A^*t}C^*Ce^{At} \, dt
\]

To compute the gramians we need to solve **2 Lyapunov equations**:

\[
\begin{align*}
AP + PA^* + BB^* &= 0, \quad P > 0 \\
A^*Q + QA + C^*C &= 0, \quad Q > 0
\end{align*}
\]

\[\Rightarrow \quad \sigma_i = \sqrt{\lambda_i(PQ)}\]

\(\sigma_i\): **Hankel singular values** of system \(\Sigma\).
SVD methods: approximation by balanced truncation

There exists basis where $P = Q = S = \text{diag}(\sigma_1, \cdots, \sigma_n)$. This is the balanced basis of the system.

In this basis partition:

$$
A = \begin{pmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{pmatrix},
B = \begin{pmatrix}
B_1
\end{pmatrix},
C = \begin{pmatrix}
C_1 \\
C_2
\end{pmatrix},
S = \begin{pmatrix}
\Sigma_1 & 0 \\
0 & \Sigma_2
\end{pmatrix}.
$$

The reduced system is obtained by balanced truncation

$$
\hat{\Sigma} = \begin{pmatrix}
A_{11} & B_1 \\
C_1 & B_1
\end{pmatrix}.
$$

$\Sigma_2$ contains the small Hankel singular values.

**Projector:** $\Pi = VW^*$ where $V = W = \begin{pmatrix} I_k \\ 0 \end{pmatrix}$. 
Properties of balanced reduction

1. **Stability is preserved**

2. **Global error bound:**

\[ \sigma_{k+1} \leq \| \Sigma - \hat{\Sigma} \|_\infty \leq 2(\sigma_{k+1} + \cdots + \sigma_n) \]

**Drawbacks**

1. **Dense** computations, matrix factorizations and inversions \(\Rightarrow\) may be ill-conditioned

2. Need **whole** transformed system in order to truncate \(\Rightarrow\) number of operations \(\mathcal{O}(n^3)\)

3. **Bottleneck:** solution of two Lyapunov equations
Approximation methods: Krylov methods

- Krylov
  - Realization
  - Interpolation
  - Lanczos
  - Arnoldi

- SVD

- Nonlinear systems
  - POD methods
  - Empirical Gramians

- Linear systems
  - Balanced truncation
  - Hankel approximation

Krylov/SVD Methods
The basic Krylov iteration

Given $A \in \mathbb{R}^{n \times n}$ and $b \in \mathbb{R}^n$, let $v_1 = \frac{b}{\|b\|}$. At the $k^{th}$ step:

$$AV_k = V_k H_k + f_k e_k^*$$

where

- $e_k \in \mathbb{R}^k$: canonical unit vector
- $V_k = [v_1 \cdots v_k] \in \mathbb{R}^{k \times k}$, $V^* V_k = I_k$
- $H_k = V_k^* A V_k \in \mathbb{R}^{k \times k}$

Computational complexity for $k$ steps: $O(n^2 k)$; storage $O(nk)$.

The Lanczos and the Arnoldi algorithms result.

The Krylov iteration involves the subspace $R_k = [b, Ab, \cdots, A^{k-1}b]$.

- **Arnoldi iteration** $\Rightarrow$ arbitrary $A \Rightarrow H_k$ upper Hessenberg.
- **Symmetric (one-sided) Lanczos iteration** $\Rightarrow$ symmetric $A = A^*$
  $\Rightarrow H_k$ tridiagonal and symmetric.
- **Two-sided Lanczos iteration** with two starting vectors $b, c$
  $\Rightarrow$ arbitrary $A \Rightarrow H_k$ tridiagonal.
Three uses of the Krylov iteration

(1) Iterative solution of $Ax = b$: approximate the solution $x$ iteratively.

(2) Iterative approximation of the eigenvalues of $A$. In this case $b$ is not fixed apriori. The eigenvalues of the projected $H_k$ approximate the dominant eigenvalues of $A$.

(3) Approximation of linear systems by moment matriching.

⇒ Item (3) is of interest in the present context.
Approximation by moment matching

Given $\dot{x}(t) = Ax(t) + Bu(t)$, $y(t) = Cx(t) + Du(t)$, expand transfer function around $s_0$:

$$G(s) = \eta_0 + \eta_1(s - s_0) + \eta_2(s - s_0)^2 + \eta_3(s - s_0)^3 + \cdots$$

Moments at $s_0$: $\eta_j$.

Find $\dot{\hat{x}}(t) = \hat{A}\hat{x}(t) + \hat{B}u(t)$, $y(t) = \hat{C}\hat{x}(t) + \hat{D}u(t)$, with

$$\hat{G}(s) = \hat{\eta}_0 + \hat{\eta}_1(s - s_0) + \hat{\eta}_2(s - s_0)^2 + \hat{\eta}_3(s - s_0)^3 + \cdots$$

such that for appropriate $s_0$ and $\ell$:

$$\eta_j = \hat{\eta}_j, \quad j = 1, 2, \ldots, \ell$$
Krylov and rational Krylov methods

• Expansion around infinity: \( \eta_j \) are the Markov parameters \( \Rightarrow \)

Problem: partial realization solved by the Krylov iteration.

• Expansion around arbitrary \( s_0 \in \mathbb{C} \): \( \eta_j \) moments \( \Rightarrow \)

Problem: rational interpolation solved by the rational Krylov iteration.

Remark. The Krylov and rational Krylov algorithms match moments without computing them. Thus moment matching methods can be implemented in a numerically efficient way.
Projectors for Krylov and rational Krylov methods

Given:

\[ \Sigma = \begin{pmatrix} E & A \\ C & D \end{pmatrix} \] by projection: \( \Pi = VW^* \), \( \Pi^2 = \Pi \) obtain

\[ \hat{\Sigma} = \begin{pmatrix} \hat{E} & \hat{A} \\ \hat{C} & \hat{D} \end{pmatrix} = \begin{pmatrix} W^*EV & W^*AV \\ CV & D \end{pmatrix}, \text{ where } k < n. \]

**Krylov:** let \( E = I \) and

\[
V = \begin{bmatrix} B, AB, \ldots, A^{k-1}B \end{bmatrix} \in \mathbb{R}^{n \times k}
\]

\[
\hat{W}^* = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{k-1} \end{bmatrix} \in \mathbb{R}^{k \times n}
\]

\[ W^* = (\hat{W}^*V)^{-1}\hat{W}^* \]

then the Markov parameters match:

\[ CA^iB = \hat{C}\hat{A}^i\hat{B} \]

**Rational Krylov:** let

\[
V = \begin{bmatrix} (\lambda_1E - A)^{-1}B \ldots (\lambda_kE - A)^{-1}B \end{bmatrix} \in \mathbb{R}^{n \times k}
\]

\[
\hat{W}^* = \begin{bmatrix} C(\lambda_{k+1}E - A)^{-1} \\ C(\lambda_{k+2}E - A)^{-1} \\ \vdots \\ C(\lambda_{2k}E - A)^{-1} \end{bmatrix} \in \mathbb{R}^{k \times n}
\]

\[ W^* = (\hat{W}^*V)^{-1}\hat{W}^* \]

then the moments of \( \hat{G} \) match those of \( G \) at \( \lambda_i \):

\[ G(\lambda_i) = D + C(\lambda_iE - A)^{-1}B = \hat{D} + \hat{C}(\lambda_i\hat{E} - \hat{A})^{-1}\hat{B} = \hat{G}(\lambda_i) \]
Properties of Krylov methods

(a) Number of operations: $O(kn^2)$ or $O(k^2n)$ vs. $O(n^3)$ ⇒ efficiency

(b) Only matrix-vector multiplications are required. No matrix factorizations and/or inversions. No need to compute transformed model and then truncate.

(c) **Drawbacks**

- global error bound?
- $\hat{\Sigma}$ may not be stable.

Q: How to choose the projection points?
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Choice of Krylov projection points: 
Passivity preserving model reduction

Passive systems:
\[ \Re \int_{-\infty}^{t} u(\tau)^* y(\tau) \, d\tau \geq 0, \quad \forall \, t \in \mathbb{R}, \, \forall \, u \in \mathcal{L}_2(\mathbb{R}). \]

Positive real rational functions:
(1) \( G(s) = D + C(sE - A)^{-1}B, \) is analytic for \( \Re(s) > 0, \)
(2) \( \Re G(s) \geq 0 \) for \( \Re(s) \geq 0, \) \( s \) not a pole of \( G(s). \)

Theorem: \( \Sigma = \left( \begin{array}{c|c} E, A & B \\ \hline C & D \end{array} \right) \) is passive \( \iff \) \( G(s) \) is positive real.

Conclusion: Positive realness of \( G(s) \) implies the existence of a spectral factorization \( G(s) + G^*(-s) = W(s)W^*(-s), \) where \( W(s) \) is stable rational and \( W(s)^{-1} \) is also stable. The spectral zeros \( \lambda_i \) of the system are the zeros of the spectral factor \( W(\lambda_i) = 0, \) \( i = 1, \cdots, n. \)
Passivity preserving model reduction

New result

- **Method**: Rational Krylov
- **Solution**: projection points = spectral zeros

Recall:
\[
V = \begin{bmatrix}
(\lambda_1 E - A)^{-1} B \\
\vdots \\
(\lambda_k E - A)^{-1} B
\end{bmatrix} \in \mathbb{R}^{n \times k}
\]

\[
W^* = \begin{bmatrix}
C(\lambda_{k+1} E - A)^{-1} \\
\vdots \\
C(\lambda_{2k} E - A)^{-1}
\end{bmatrix} \in \mathbb{R}^{k \times n}
\]

**Main result.** If \( V, W \) are defined as above, where \( \lambda_1, \cdots, \lambda_k \) are spectral zeros, and in addition \( \lambda_{k+i} = -\lambda_i^* \), the reduced system satisfies:

(i) the interpolation constraints,
(ii) it is stable, and
(iii) it is passive.
Spectral zero interpolation preserving passivity

Hamiltonian EVD & projection

Hamiltonian eigenvalue problem

\[
\begin{bmatrix}
A & 0 & B \\
0 & -A^* & -C^* \\
C & B^* & \Delta^{-1}
\end{bmatrix}
\begin{bmatrix}
X \\
Y \\
Z
\end{bmatrix}
=
\begin{bmatrix}
E & 0 & 0 \\
0 & E^* & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
X \\
Y \\
Z
\end{bmatrix}
\Lambda
\]

The generalized eigenvalues $\Lambda$ are the spectral zeros of $\Sigma$

Partition eigenvectors

\[
\begin{bmatrix}
X \\
Y \\
Z
\end{bmatrix}
=
\begin{bmatrix}
X_- & X_+ \\
Y_- & Y_+ \\
Z_- & Z_+
\end{bmatrix},
\Lambda =
\begin{bmatrix}
\Lambda_- \\
\Lambda_+ \\
\pm \infty
\end{bmatrix}
\]

$\Lambda_-$ are the stable spectral zeros

Reduce by projection

- $V = X_-$, $W = Y_-$
- $\hat{E} = W^*EV$, $\hat{A} = W^*AV$, $\hat{B} = W^*B$, $\hat{C} = CV$, $\hat{D} = D$
Dominant spectral zeros – SADPA

SADPA

What is a good choice of \( k \) spectral zeros out of \( n \)?

- **Dominance criterion:** Spectral zero \( s_j \) is **dominant** if: \( \left| \frac{R_j}{\Re(s_j)} \right| \) is large.

- Efficient computation for large scale systems: we compute the \( k \ll n \) **most dominant** eigenmodes of the Hamiltonian pencil.

- **SADPA** (Subspace Accelerated Dominant Pole Algorithm) solves this **iteratively**.

**Conclusion:**
Passivity preserving model reduction becomes a **structured eigenvalue problem**.
Choice of Krylov projection points:
Optimal $\mathcal{H}_2$ model reduction

The $\mathcal{H}_2$ norm of a (scalar) system is:

$$\|\Sigma\|_{\mathcal{H}_2} = \left(\int_{-\infty}^{+\infty} h^2(t) dt\right)^{1/2}$$

**Goal:** construct a **Krylov projection** such that

$$\Sigma_k = \arg\min_{\text{deg}(\hat{\Sigma}) = r, \hat{\Sigma} : \text{stable}} \|\Sigma - \hat{\Sigma}\|_{\mathcal{H}_2}.$$ 

That is, find a **Krylov projection** $\Pi = VW^*$, $V, W \in \mathbb{R}^{n \times k}$, $W^*V = I_k$, such that:

$$\hat{A} = W^*AV, \quad \hat{B} = W^*B, \quad \hat{C} = CV$$
Necessary optimality conditions & resulting algorithm

Let \((\hat{A}, \hat{B}, \hat{C})\) solve the optimal \(\mathcal{H}_2\) problem and let \(\hat{\lambda}_i\) denote the eigenvalues of \(\hat{A}\). The necessary optimality conditions are

\[
\mathbf{G}(\hat{-\lambda}_i^*) = \hat{\mathbf{G}}(\hat{-\lambda}_i^*) \quad \text{and} \quad \frac{d}{ds} \mathbf{G}(s) \bigg|_{s=-\hat{\lambda}_i^*} = \frac{d}{ds} \hat{\mathbf{G}}(s) \bigg|_{s=-\hat{\lambda}_i^*}
\]

Thus the reduced system has to match the first two moments of the original system at the *mirror images* of the eigenvalues of \(\hat{A}\). The proposed algorithm produces such a reduced order system.

1. Make an initial selection of \(\sigma_i\), for \(i = 1, \ldots, k\)
2. \(W = [(\sigma_1 I - A^*)^{-1} C^*, \ldots, (\sigma_k I - A^*)^{-1} C^*]\)
3. \(V = [(\sigma_1 I - A)^{-1} B, \ldots, (\sigma_k I - A)^{-1} B]\)
4. while (not converged)
   - \(\hat{A} = (W^* V)^{-1} W^* AV\)
   - \(\sigma_i \leftarrow -\lambda_i(\hat{A}) + \text{Newton correction}, \; i = 1, \ldots, k\)
   - \(W = [(\sigma_1 I - A^*)^{-1} C^*, \ldots, (\sigma_k I - A^*)^{-1} C^*]\)
   - \(V = [(\sigma_1 I - A)^{-1} B, \ldots, (\sigma_k I - A)^{-1} B]\)
5. \(\hat{A} = (W^* V)^{-1} W^* AV, \; \hat{B} = (W^* V)^{-1} W^* B, \; \hat{C} = CV\)
Example: Transmission line (E singular)

oscillatory frequency response

\[ C = 10^{-8} \, F, \quad L = 10^{-6} \, H, \quad R_C = 10^8 \, \Omega, \]
\[ R_L = 10^{-2} \, \Omega, \quad R = 10\, \Omega \]

non-oscillatory frequency response

\[ C = 10^{-8} \, F, \quad L = 10^{-6} \, H, \quad R_C = 10^8 \, \Omega \]
\[ R_L = 1\, \Omega, \quad R = 1\, \Omega \]
Choice of projection points

Optimal $H_2$ model reduction

### Plots

#### Oscillatory system

Hankel singular values and Positive real singular values

- $\text{dim} = 101$

#### Non-oscillatory system

Hankel singular values and Positive real singular values

- $\text{dim} = 101$

Error plots, Positive Real Balancing, Prima and Dominant SZM

- $n = 155$
- $\text{dim} = 101$
- $k = 11$

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Thanos Antoulas (Rice University)  
Model reduction of large-scale systems
### $H_\infty$ and $H_2$ error norms

#### Relative norms of the error systems

<table>
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<tr>
<th>Reduction Method</th>
<th>Oscillatory</th>
<th>Non-oscillatory</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n = 155$, $dim = 101$, $k = 11$</td>
<td>$H_\infty$ $H_2$</td>
<td>$H_\infty$ $H_2$</td>
</tr>
<tr>
<td>PRIMA</td>
<td>1.4775</td>
<td>-</td>
</tr>
<tr>
<td>Spectral Zero Method (SZM with SADPA)</td>
<td>0.9628</td>
<td>0.841</td>
</tr>
<tr>
<td>SZM with SADPA ($interp = 31$) followed by PRBT</td>
<td>0.9617</td>
<td>0.8144</td>
</tr>
<tr>
<td>Optimal $H_2$</td>
<td>0.5943</td>
<td>0.4621</td>
</tr>
<tr>
<td>Balanced truncation (BT)</td>
<td>0.9393</td>
<td>0.6466</td>
</tr>
<tr>
<td>Riccati Balanced Truncation (PRBT)</td>
<td>0.9617</td>
<td>0.8164</td>
</tr>
</tbody>
</table>
Approximation methods: Summary

**Krylov**
- Realization
- Interpolation
- Lanczos
- Arnoldi

**SVD**

**Nonlinear systems**
- POD methods
- Empirical Gramians

**Linear systems**
- Balanced truncation
- Hankel approximation

**Krylov/SVD Methods**
- **Properties**
  - Numerical efficiency
  - $n \gg 10^3$

**Properties**
- Stability
- Error bound
- $n \approx 10^3$
Outline

1. Overview
2. Motivation
3. Approximation methods
   - SVD-based methods
   - Krylov-based methods
4. Choice of projection points
   - Passivity preserving model reduction
   - Optimal $H_2$ model reduction
5. Model reduction from measurements
   - S-parameters
   - Classical realization theory
   - Finite data points: the Loewner matrix
   - Tangential interpolation: $\mathbb{L}$ & $\sigma\mathbb{L}$
   - Recursive Loewner-matrix framework
6. Conclusions
Recall: the big picture

Physical System

Modeling

Data

and/or

discretization

ODEs

Model reduction

PDEs

Simulation

Control

reduced # of ODEs
A motivation: electronic systems

- Growth in communications and networking and demand for high data bandwidth requires streamlining of the simulation of entire complex systems from chips to packages to boards, etc.
- Thus in circuit simulation, signal integrity (lack of signal distortion) of high speed electronic designs require that interconnect models be valid over a wide bandwidth.

An important tool: **S-parameters**

- They represent a component as a black box. Accurate simulations require accurate component models.
- In high frequencies S-parameters are important because wave phenomena become dominant.
- Advantages: $0 \leq |S| \leq 1$ and can be measured using VNAs (Vector Network Analyzers).
Scattering or S-parameters

Given a system in I/O representation: \( y(s) = H(s)u(s) \),
the associated **S-parameter representation** is

\[
\bar{y}(s) = S(s)\bar{u}(s) = \left[ H(s) + I \right]\left[ H(s) - I \right]^{-1} \bar{u}(s),
\]

where

\[
\bar{y} = \frac{1}{2}(y + u)
\]

are the *transmitted waves* and,

\[
\bar{u} = \frac{1}{2}(y - u)
\]

are the *reflected waves*.

**S-parameter measurements.**

\( S(j\omega_k) \): samples of the frequency response of the S-parameter system representation.
Measurement of S-parameters

VNA (Vector Network Analyzer) – Magnitude of S-parameters for 2 ports
Model construction from data (at infinity):
Classical realization

Given $h_t \in \mathbb{R}^{p \times m}$, $t = 1, 2, \cdots$, find $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{p \times n}$, such that

$$h_t = CA^{t-1}B, \quad t > 0$$

**Main tool:** Hankel matrix

$$H = \begin{bmatrix}
    h_1 & h_2 & h_3 & \cdots \\
    h_2 & h_3 & h_4 & \cdots \\
    h_3 & h_4 & h_5 & \cdots \\
    \vdots & \vdots & \vdots & \ddots
\end{bmatrix} = \begin{bmatrix}
    C \\
    CA \\
    CA^2 \\
    \vdots
\end{bmatrix} \mathcal{R} = \begin{bmatrix}
    B & AB & A^2B & \cdots \\
    \vdots & \ddots & \ddots & \ddots
\end{bmatrix}$$
Classical realization

**Solvability** $\iff \text{rank } \mathcal{H} = n < \infty$

**Solution:**
- Find $\Delta \in \mathbb{R}^{n \times n}$: $\det \Delta \neq 0$.
- Let $\sigma \Delta \in \mathbb{R}^{n \times n}$: shifted matrix.
- Let $\Gamma \in \mathbb{R}^{n \times m}$: first $m$ columns, while $\Lambda \in \mathbb{R}^{p \times n}$: the first $p$ rows.

Then

$$A = \Delta^{-1} \sigma \Delta, \quad B = \Delta^{-1} \Gamma, \quad C = \Lambda.$$  

**Consequence.** In terms of the data:

$$\mathbf{H}(s) = \Lambda (s \Delta - \sigma \Delta)^{-1} \Gamma$$
Model construction from data at finite points: Interpolation

Assume for simplicity that the given data are scalar:

\[(s_i, \phi_i), \ i = 1, 2, \cdots, N, \ s_i \neq s_j, \ i \neq j\]

Find \(H(s) = \frac{n(s)}{d(s)}\) such that \(H(s_i) = \phi_i, \ i = 1, 2, \cdots, N\), and \(n, d\): coprime polynomials.

A solution always exists (e.g. Lagrange interpolating polynomial).

Additional constraints for \(H\): minimality, stability, bounded realness etc.

**Main tool:** Loewner matrix. Divide the data in disjoint sets:

\[(\lambda_i, w_i), \ i = 1, 2, \cdots, k, (\mu_j, v_j), j = 1, 2, \cdots, q, k + q = N:\]

\[
L_i = \begin{bmatrix}
\frac{v_1-w_1}{\mu_1-\lambda_1} & \cdots & \frac{v_1-w_k}{\mu_1-\lambda_k} \\
\vdots & \ddots & \vdots \\
\frac{v_q-w_1}{\mu_q-\lambda_1} & \cdots & \frac{v_q-w_k}{\mu_q-\lambda_k}
\end{bmatrix} \in \mathbb{C}^{q \times k}
\]
Main result (1986). The rank of $\mathbb{L}$ encodes the information about the minimal degree interpolants: $\text{rank } \mathbb{L}$ or $N - \text{rank } \mathbb{L}$.

Remarks.
(a) In this framework the strict properness assumption has been dropped. Thus rational functions with polynomial part can be recovered from input-output data.
(b) The construction of interpolants will be deferred until later.
(c) If $H(s) = C(sI - A)^{-1}B + D$, then

$$\mathbb{L} = -\begin{bmatrix} C(\lambda_1 I - A)^{-1} \\ C(\lambda_2 I - A)^{-1} \\ \vdots \\ C(\lambda_k I - A)^{-1} \end{bmatrix} \begin{bmatrix} (\mu_1 I - A)^{-1}B & \cdots & (\mu_q I - A)^{-1}B \end{bmatrix}$$
Scalar interpolation – multiple points

**Special case.** single point with multiplicity: \((s_0; \phi_0, \phi_1, \cdots, \phi_{N-1})\), i.e. the value of the function and that of a number of derivatives is provided. The **Loewner matrix** becomes:

\[
\mathbb{L} = \begin{bmatrix}
\frac{\phi_1}{1!} & \frac{\phi_2}{2!} & \frac{\phi_3}{3!} & \frac{\phi_4}{4!} & \cdots \\
\frac{\phi_2}{2!} & \frac{\phi_3}{3!} & \frac{\phi_4}{4!} & \cdots \\
\frac{\phi_3}{3!} & \frac{\phi_4}{4!} & \cdots \\
\frac{\phi_4}{4!} & \cdots \\
\vdots & \ddots & \ddots \\
\vdots & \ddots & \ddots
\end{bmatrix}
\]

\[\Rightarrow\] **Hankel matrix**

Thus the **Loewner matrix** generalizes the **Hankel matrix** when general interpolation replaces realization.
General framework – tangential interpolation

Given: • right data: \((\lambda_i; r_i, w_i), i = 1, \cdots, k\)

• left data: \((\mu_j; \ell_j, v_j), j = 1, \cdots, q\).

We assume for simplicity that all points are distinct.

**Problem**: Find rational \(p \times m\) matrices \(H(s)\), such that

\[
H(\lambda_i) r_i = w_i \quad \ell_j H(\mu_j) = v_j
\]

Right data:

\[
\Lambda = \begin{bmatrix}
\lambda_1 \\
\vdots \\
\lambda_k
\end{bmatrix} \in \mathbb{C}^{k \times k},
\]

\[
R = [r_1 \ r_2 \ \cdots \ r_k] \in \mathbb{C}^{m \times k},
\]

\[
W = [w_1 \ w_2 \ \cdots \ w_k] \in \mathbb{C}^{p \times k}
\]

Left data:

\[
M = \begin{bmatrix}
\mu_1 \\
\vdots \\
\mu_q
\end{bmatrix} \in \mathbb{C}^{q \times q},
\]

\[
L = \begin{bmatrix}
\ell_1 \\
\vdots \\
\ell_q
\end{bmatrix} \in \mathbb{C}^{q \times p},
\]

\[
V = \begin{bmatrix}
v_1 \\
\vdots \\
v_q
\end{bmatrix} \in \mathbb{C}^{q \times m}
\]
General framework – tangential interpolation

Input-output data. The Loewner matrix is:

\[
\mathcal{L} = \begin{bmatrix}
\frac{v_1 r_1 - \ell_1 w_1}{\mu_1 - \lambda_1} & \ldots & \frac{v_1 r_k - \ell_1 w_k}{\mu_1 - \lambda_k} \\
\vdots & \ddots & \vdots \\
\frac{v_q r_1 - \ell_q w_1}{\mu_q - \lambda_1} & \ldots & \frac{v_q r_k - \ell_q w_k}{\mu_q - \lambda_k}
\end{bmatrix} \in \mathbb{C}^{q \times k}
\]

Recall:

\[
H(\lambda_i) r_i = w_i, \quad \ell_j H(\mu_j) = v_j
\]

Therefore \( \mathcal{L} \) satisfies the Sylvester equation

\[
\mathcal{L} \Lambda - M \mathcal{L} = V R - L W
\]
General framework – tangential interpolation

**State space data.** Suppose that \( E, A, B, C \) of minimal degree \( n \) are given such that \( H(s) = C(sE - A)^{-1}B \).

Let \( X, Y \) satisfy the following **Sylvester equations**

\[
EX - AX = BR \quad \text{and} \quad MYE - YA = LC
\]

If the generalized eigenvalues of \((E, A)\) are distinct from \( \lambda_i \) and \( \mu_j \), \( X, Y \) are unique solutions of these equations. Actually

\[
x_i = (\lambda_i E - A)^{-1}Br_i \implies X: \text{generalized reachability matrix}
\]

\[
y_j = \ell_j C(\mu_j E - A)^{-1} \implies Y: \text{generalized observability matrix}.
\]

\[
\Rightarrow \mathbb{L} = -YEX
\]
Construction of Interpolants

Suggested construction procedure for an interpolant of McMillan degree $n$:

1. Factor $\mathbb{L} = -YEX$, so that $E$ has rank $n$.
2. Construct $A, B, C$ to satisfy the Sylvester equations above.
3. Define $D := w_j - C(s_jE - A)^{-1}B$.

Steps 2 and 3 are easy; the first step is problematic: how do we choose $E$?

- If the system is proper, then $\text{size}(E) = \text{rank}(E)$, and we could use, for example, the SVD to factor $\mathbb{L} \Rightarrow$ proper systems are easy.
- If the system is singular, then $\text{size}(E) > \text{rank}(E)$, and we’re stuck.

Solution: use the \textbf{shifted Loewner matrix} $\sigma\mathbb{L}$
The shifted Loewner matrix

- The shifted Loewner matrix, $\sigma L$, is the Loewner matrix associated to $sH(s)$.

$$
\sigma L = \begin{bmatrix}
\frac{\mu_1 v_1 r_1 - \ell_1 w_1 \lambda_1}{\mu_1 - \lambda_1} & \ldots & \frac{\mu_1 v_1 r_k - \ell_1 w_k \lambda_k}{\mu_1 - \lambda_k} \\
\frac{\mu q v_q r_1 - \ell q w_1 \lambda_1}{\mu q - \lambda_1} & \ldots & \frac{\mu q v_q r_k - \ell q w_k \lambda_k}{\mu q - \lambda_k}
\end{bmatrix} \in \mathbb{C}^{q \times k}
$$

- $\sigma L$ satisfies the Sylvester equation

$$
\sigma L \Lambda - M \sigma L = VR \Lambda - MLW
$$

- $\sigma L$ can be factored as

$$
\Rightarrow \quad \sigma L = -YAX
$$
Construction of Interpolants (Models)

Assume that \( k = \ell \), and let

\[
\det (xL - \sigma L) \neq 0, \quad x \in \{\lambda_i\} \cup \{\mu_j\}
\]

Then

\[
E = -L, \quad A = -\sigma L, \quad B = V, \quad C = W
\]

is a minimal realization of an interpolant of the data, i.e., the function

\[
H(s) = W(\sigma L - sL)^{-1}V
\]

interpolates the data.

**Proof.** Multiplying the first equation by \( s \) and subtracting it from the second we get

\[
(\sigma L - sL)\Lambda - M(\sigma L - sL) = LW(\Lambda - sI) - (M - sI)VR.
\]

Multiplying this equation by \( e_i \) on the right and setting \( s = \lambda_i \), we obtain

\[
(\lambda_iI - M)(\sigma L - \lambda_iL)e_i = (\lambda_iI - M)Vr_i \quad \Rightarrow \quad (\lambda_iL - \sigma L)e_i = Vr_i \quad \Rightarrow \quad We_i = W(\lambda_iL - \sigma L)^{-1}V
\]

Therefore \( w_i = H(\lambda_i)r_i \). This proves the right tangential interpolation property. To prove the left tangential interpolation property, we multiply the above equation by \( e_j^* \) on the left and set \( s = \mu_j \):

\[
e_j^*(\sigma L - \mu_jL)(\Lambda - \mu_jI) = e_j^*LW(\mu_jI - \Lambda) \quad \Rightarrow \quad e_j^*(\sigma L - \mu_jL) = \ell_jW \quad \Rightarrow \quad e_j^*V = \ell_jW(\sigma L - \mu_jL)^{-1}V
\]

Therefore \( v_j = \ell_jH(\mu_j) \).
Construction of interpolants:
New procedure

Main assumption:

\[
\text{rank}(xL - \sigma L) = \text{rank}(L \sigma L) = \text{rank}\left(\begin{pmatrix} L \\ \sigma L \end{pmatrix}\right) = k, \ x \in \{\lambda_i\} \cup \{\mu_j\}
\]

Then for some \( x \in \{\lambda_i\} \cup \{\mu_j\} \), we compute the SVD

\[
xL - \sigma L = Y\Sigma X
\]

with \( \text{rank}(xL - \sigma L) = \text{rank}(\Sigma) = \text{size}(\Sigma) = k \), \( Y \in \mathbb{C}^{\nu \times k} \), \( X \in \mathbb{C}^{k \times \rho} \).

**Theorem.** A realization \([E, A, B, C]\), of an interpolant is given as follows:

\[
\begin{align*}
E &= -Y^*LX^* \\
A &= -Y^*\sigma L X^* \\
B &= Y^*V \\
C &= WX^*
\end{align*}
\]

**Remark.** The system \([E, A, B, C]\) can now be further reduced using any of the usual reduction methods.
Example: mechanical system

Mechanical example: Stykel, Mehrmann

\[
\begin{align*}
\dot{p}(t) &= v(t) \\
M \dot{v}(t) &= Kp(t) + Dv(t) - G^* \lambda(t) + B_2 u(t) \\
0 &= Gp(t) \\
y(t) &= C_1 p(t)
\end{align*}
\]

Measurements: 500 frequency response data between \([-2i, +2i]\).
Mechanical system: plots

**Left:** Frequency responses of original system and approximants (orders 2, 6, 10, 14, 18)

**Right:** Frequency responses of error systems
Example: Four-pole band-pass filter

- 1000 measurements between 40 and 120 GHz; S-parameters $2 \times 2$, MIMO (approximate) interpolation $\Rightarrow L, \sigma L \in \mathbb{R}^{2000 \times 2000}$.

The singular values of $L$, $\sigma L$

The $S(1, 1)$ and $S(1, 2)$ parameter data

17-th order model
Recursive Loewner-matrix framework

Consider the (right and left) interpolation data defined by the matrices

\[
\begin{bmatrix}
R & \in & \mathbb{C}^{m \times k},
W & \in & \mathbb{C}^{p \times k},
\Lambda & \in & \mathbb{C}^{k \times k}
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
L & \in & \mathbb{C}^{\ell \times p},
V & \in & \mathbb{C}^{\ell \times m},
M & \in & \mathbb{C}^{\ell \times \ell}
\end{bmatrix},
\]

as well as the associated \textit{Loewner matrix} $L$ and \textit{shifted Lowener matrix} $\sigma L$, both of size $\ell \times k$, which satisfy the Sylvester equations

\[
\begin{align*}
L\Lambda - ML &= LW - VR = (L V) \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} R \\ W \end{pmatrix}, \\
\sigma L\Lambda - M\sigma L &= LW\Lambda - MVR = (L MV) \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} R \\ W\Lambda \end{pmatrix}.
\end{align*}
\]

We now define the $(p + m) \times (p + m)$ rational matrices:

\[
\Theta(s) = \begin{bmatrix} I_p & 0 \\ 0 & I_m \end{bmatrix} + \begin{bmatrix} W \\ -R \end{bmatrix} (sL - L\Lambda)^{-1} \begin{bmatrix} L \\ V \end{bmatrix} = \begin{pmatrix} \Theta_{11}(s) & \Theta_{12}(s) \\ \Theta_{21}(s) & \Theta_{22}(s) \end{pmatrix},
\]

\[
\bar{\Theta}(s) = \Theta^{-1}(s) = \begin{bmatrix} I_p & 0 \\ 0 & I_m \end{bmatrix} + \begin{bmatrix} -W \\ R \end{bmatrix} (sL - M\Lambda)^{-1} \begin{bmatrix} L \\ V \end{bmatrix} = \begin{pmatrix} \bar{\Theta}_{11}(s) & \bar{\Theta}_{12}(s) \\ \bar{\Theta}_{21}(s) & \bar{\Theta}_{22}(s) \end{pmatrix}.
\]
Recursive interpolation

These are the *generating matrices* of the interpolation problem, as all interpolants can be obtained from these quantities. First, we notice that the interpolation conditions are satisfied.

**Lemma**

\[ [L_j \ V_j] \Theta(\mu_j) = 0_{\ell \times (p+m)}, \text{ for all } j, \text{ and } \bar{\Theta}(\lambda_k) \left( \begin{array}{c} \bar{W}_k \\ R_k \end{array} \right) = 0_{(p+m) \times k}, \text{ for all } k. \]

Next, we assert that all interpolants can be obtained as linear matrix fractions constructed from the entries of \( \Theta \) and \( \bar{\Theta} \).

**Theorem**

The above result implies that the rational matrix \( \Psi \) is an interpolant for any \( \Gamma(s) \), where:

\[
\Psi(s) = [\Theta_{11}(s)\Gamma(s) + \Theta_{12}(s)][\Theta_{21}(s)\Gamma(s) + \Theta_{22}(s)]^{-1} = [\bar{\Theta}_{11}(s) - \Gamma(s)\bar{\Theta}_{21}(s)]^{-1}[\bar{\Theta}_{12}(s) - \Gamma(s)\bar{\Theta}_{22}(s)],
\]

are right, left coprime factorizations, respectively.
Cascade representation of recursive interpolation

Feedback interpretation of the parametrization of all solutions of the rational interpolation problem

\[ \begin{align*}
\Theta &
\end{align*} \]

Cascade representation of the recursive interpolation problem.
Recursive Loewner and shifted Loewner matrices

For the recursive procedure, the error quantities at each step are the key, and are computed as follows:

\[
\begin{bmatrix}
L_{k,e} & V_{k,e}
\end{bmatrix} = \begin{bmatrix}
L_k & V_k
\end{bmatrix} \Theta_{k-1}(\mu_k) \quad \text{and} \quad \begin{bmatrix}
-W_{k,e} \\
R_{k,e}
\end{bmatrix} = \hat{\Theta}_{k-1}(\lambda_k) \begin{bmatrix}
-W_k \\
R_k
\end{bmatrix}.
\]

It follows that the recursive quantities for 3 stages are:

\[
w_{e} = [w_{e1} \ w_{e2} \ w_{e3}], \quad L_{e} = \begin{bmatrix}
L_{1e} & L_{2e} & L_{3e}
\end{bmatrix}, \quad \sigma_{L_{e}} = \begin{bmatrix}
\sigma L_{1e} & L_{1e}W_{2e} & L_{1e}W_{3e} \\
V_{2e}R_{1e} & \sigma L_{2e} & L_{2e}W_{3e} \\
V_{3e}R_{2e} & V_{3e}R_{3e} & \sigma L_{3e}
\end{bmatrix}, \quad V_{e} = \begin{bmatrix}
v_{1e} \\
v_{2e} \\
v_{3e}
\end{bmatrix}.
\]

The resulting generating system is

\[
\Theta(s) = \begin{bmatrix}
I_p & 0 \\
0 & I_m
\end{bmatrix} + \begin{bmatrix}
W_{e}
\end{bmatrix} (S_{L_{e}} - \sigma_{L_{e}} + V_{e}R_{e})^{-1} \begin{bmatrix}
L_{e} & V_{e}
\end{bmatrix}
\]

The above procedure recursively constructs an L-D-U factorization of the Loewner matrix.
Example: delay system

\[
\dot{E}x(t) = A_0 x(t) + A_1 x(t - \tau) + Bu(t), \quad y(t) = Cx(t),
\]

where \( E, A_0, A_1 \) are 500 \( \times \) 500 and \( B, C^* \) are 500-vectors.

**Procedure:** compute 1000 frequency response samples. Then apply recursive/adaptive Loewner-framework procedure. (Blue: original, red: approximants.)

35-th order adaptively constructed model; \( H_\infty \) norm of error: 0.008.

50-th order non-adaptively constructed model; \( H_\infty \) norm of error: 0.180.
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6. Conclusions
Summary and conclusions

- Given input/output data, we can construct with no computation, a high order system in generalized state space form $\Rightarrow$

    let the data reveal the underlying system

- This system is such that $(\mathbb{L}, \sigma\mathbb{L})$ is a singular pencil.

- To address this issue:
  1. SVD of $\mathbb{L}$ $(\sigma\mathbb{L})$.
  2. Recursive (and adaptive) procedure.

- Approach is the natural way to construct models and reduced models from data as it does not require (force) the inversion of $E$. 
References

- **Passivity preserving model reduction**

- **Optimal $H_2$ model reduction**

- **Model reduction from data**

- **General reference:** Antoulas, SIAM (2005)