Positive descriptor systems
Linear Systems Workshop
Beer-Sheva, Israel

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The most efficient way to control a car to a destination:

- Pedal to the metal!
- Full breaking!
- Bang-bang control
Challenge: Control of positive systems

Medicine:
▷ e.g. concentration of a drug in a human body

Bio
▷ e.g. number of cells, bacteria, species

\[
F(t, x, \dot{x}, u) = 0, \quad x(t_0) = x_0
\]

\[
y = G(x, u)
\]

Linearization along constant trajectories leads to:

\[
E \dot{x} = Ax + Bu, \quad x(t_0) = x_0
\]

Positivity requires non-negative solution \(x \geq 0\) and output \(y \geq 0\).

First results

• If \((E, A)\) is regular, then

\[
\hat{E} = (\lambda E - A)^{-1} E
\]

and

\[
\hat{A} = (\lambda E - A)^{-1} A
\]

commute.

• Explicit solution representation for \((E, A) = \nu\):

\[
x(t) = e^{\hat{E}D\hat{A}t} e^{\hat{E}D\hat{Ev}} + \int_{t_0}^{t} e^{\hat{E}D\hat{A}(t-\tau)} \hat{E}D\hat{Bu}(
\]

\[
≥ \nu - 1 \sum_{i=0}^{\nu} (\hat{E}\hat{A}D)^i \hat{A}D\hat{Bu}(i)(t).
\]

• New Perron-Frobenius Theory for matrix pairs: Let \((E, A)\) regular of \((E, A) = \nu\).

\[
\hat{E}D\hat{A} \geq 0,
\]

then \(\rho_f(E, A)\) is an eigenvalue of \((E, A)\) and there exists a corresponding non-negative eigenvector.

• Stability results for ODEs carry over to DAEs (for general index \(\nu\)).

• Doubly non-negative solution of generalized projected Lyapunov equations.

• An SVD algorithm that identifies meta-stable states of Markov chains.


E. Virnik. Stability analysis of positive descriptor systems. Submitted for publication.


Further goals

• Complete characterization of positive reachability, controllability, observability.

• Positivity preserving methods for: optimal control; preconditioning; discretization; model reduction.

Cooperations

Internal: A1; A4; A6; D13; External: D. B. Szyld (Temple University); D. Fritzsche (Universität Wuppertal); H. Schneider (Madison); R. Bru (Universidad Politécnica de Valencia); R. Shorten (Hamilton Institute).
We consider positive descriptor systems:

\[
F(t, x, \dot{x}, u) = 0, \quad x(t_0) = x_0, \\
y = G(x, u).
\]

Linearisation along constant trajectories leads to:

\[
E\dot{x} = Ax + Bu, \quad x(t_0) = x_0, \\
y = Cx + Du.
\]

**Positivity**

Nonnegative state \(x\), input \(u\), output \(y\) for all \(t\).
Definition

The matrix pair $(E, A)$ is called

- **regular** if $E$ and $A$ are square and $\det(\lambda E - A) \neq 0$ for some $\lambda \in \mathbb{R}$.
- **singular** otherwise.
**Definition**

The matrix pair \((E, A)\) is called

- **regular** if \(E\) and \(A\) are square and \(\det(\lambda E - A) \neq 0\) for some \(\lambda \in \mathbb{R}\).
- **singular** otherwise.

**Lemma (Commuting matrices [Campbell ’80])**

- \((E, A)\) regular;
- \(\hat{\lambda}\) such that \(\det(\hat{\lambda}E - A) \neq 0\).

\[\Rightarrow \hat{E} := (\hat{\lambda}E - A)^{-1}E \quad \text{and} \quad \hat{A} := (\hat{\lambda}E - A)^{-1}A \text{ commute.}\]
Weierstraß canonical form

Let \((E, A)\) regular. Then, there exist regular \(W, T \in \mathbb{R}^{n \times n}\) such that

\[
(E, A) = \begin{pmatrix}
W \begin{bmatrix}
I & 0 \\
0 & N
\end{bmatrix} T, W \begin{bmatrix}
J & 0 \\
0 & I
\end{bmatrix} T
\end{pmatrix}, \quad N \text{ nilpotent.}
\]

Definition

The index of the DAE is defined by the index of nilpotency of \(N\).

Definition

The spectral projector is defined by \(P_r := T^{-1} \begin{bmatrix}
I & 0 \\
0 & 0
\end{bmatrix} T\).
Eigenvalues of matrix pairs

Definition (Eigenvalues and -vectors of a matrix pair)

- $\lambda \in \mathbb{C}$ is (finite) eigenvalue of $(E, A)$ if $\det(\lambda E - A) = 0$,
- $v \in \mathbb{C}^n$ is a corresponding eigenvector if $(\lambda E - A)v = 0$,
- $v \in \mathbb{C}^n$ is eigenvector corresponding to $\infty$ if $Ev = 0$.

Definition (Finite deflating subspace)

The subspace spanned by all Jordan chains corresponding to finite eigenvalues is called finite deflating subspace.

$P_r$ is the spectral projector onto the finite deflating subspace along the subspace spanned by all Jordan chains corresponding to $\infty$. 
Definition

The index of $A$ is the least nonnegative integer $\nu$ such that $\ker(A^{\nu+1}) = \ker(A^\nu)$. The Drazin inverse of $A$ is the unique matrix $A^D$, which satisfies

$$A^D A = AA^D, \quad A^D AA^D = A^D, \quad A^{\nu+1} A^D = A^\nu$$
**Drazin generalised inverse**

**Definition**

The index of \( A \) is the least nonnegative integer \( \nu \) such that \( \ker(A^{\nu+1}) = \ker(A^\nu) \). The Drazin inverse of \( A \) is the unique matrix \( A^D \), which satisfies

\[
A^D A = AA^D \quad A^D AA^D = A^D \quad A^{\nu+1} A^D = A^\nu
\]

**Example**

Let \( A = \begin{bmatrix} C & 0 \\ 0 & N \end{bmatrix} \), where \( C \) is regular and \( N \) is nilpotent, then

\[
A^D = \begin{bmatrix} C^{-1} & 0 \\ 0 & 0 \end{bmatrix}.
\]
For \((E, A)\) regular, \(\hat{E} := (\lambda E - A)^{-1} E, \hat{A} := (\lambda E - A)^{-1} A, u \in C^\nu\) with \(\nu = \text{ind}(E)\), we have that

\[\begin{align*}
\text{for some } \nu \in \mathbb{C}^n, \text{ every solution to } E\dot{x} = Ax + Bu \text{ has the form:} \\
x(t) &= e^{\hat{E}D\hat{A}t} \hat{E}D\hat{A}v + \int_0^t e^{\hat{E}D\hat{A}(t-\tau)} \hat{E}D\hat{B}u(\tau)d\tau - \\
&\quad -(I - \hat{E}D\hat{E}) \sum_{i=0}^{\nu-1} (\hat{E}\hat{A}D)^i \hat{A}D\hat{B}u^{(i)}(t); \\
\text{we have a (unique) solution iff there exists a } v \in \mathbb{C}^n \text{ such that} \\
x_0 &= \hat{E}D\hat{E}v - (I - \hat{E}D\hat{E}) \sum_{i=0}^{\nu-1} (\hat{E}\hat{A}D)^i \hat{A}D\hat{B}u^{(i)}(0).
\end{align*}\]
Definition (Lyapunov stability)

The solution $x(t) \equiv 0$ of $E \dot{x} = Ax$ is called Lyapunov stable, if for all $\epsilon > 0$ there exists $\delta > 0$, such that $\|x(t, x_0)\| < \epsilon$ for all $t \geq 0$ and for all $x_0 \in \text{im } \hat{E} D \hat{E}$ with $\|x_0\| < \delta$. 

Definition (Asymptotic stability)

The solution $x(t) \equiv 0$ is called asymptotically stable, if it is
1. Lyapunov stable
2. there exists $\delta > 0$, such that $x(t, x_0) \to 0$ for $t \to \infty$ and for all $x_0 \in \text{im } \hat{E} D \hat{E}$ with $\|x_0\| < \delta$. 
Definition (Lyapunov stability)

The solution $x(t) \equiv 0$ of $\dot{E}x = Ax$ is called **Lyapunov stable**, if for all $\epsilon > 0$ there exists $\delta > 0$, such that $\|x(t, x_0)\| < \epsilon$ for all $t \geq 0$ and for all $x_0 \in \text{im} \hat{E} \hat{D} \hat{E}$ with $\|x_0\| < \delta$.

Definition (Asymptotic stability)

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Equivalent characterisation

The system

\[ \dot{x} = Ax, \quad x(0) = x_0. \]

is (asymptotically) stable \iff all eigenvalues of \( A \) are in \( \mathbb{C}^- \).

Definition

The matrix \( A \) is called \textit{c-stable}. 
### Equivalent characterisation

The system

\[ E \dot{x} = Ax, \quad x(0) = x_0. \]

is (asymptotically) stable \( \iff \) all finite eigenvalues of \((E, A)\) are in \(\mathbb{C}^{-}\).

### Definition

The matrix pair \((E, A)\) is called \textbf{c-stable}.
### Characterisation [Luenberger ’79]

\[
\dot{x} = Ax + Bu \text{ is positive iff } A \text{ is a } -Z\text{-matrix and } B \geq 0.
\]

### Stability [Farina/Rinaldi ’00]

Relaxed conditions due to Perron-Frobenius Theory:

- **dominant eigenvalue is real**
  - system is asymptotically stable iff all real eigenvalues are in \( \mathbb{C}_- \)
- **diagonal** positive definite matrix is Lyapunov function; useful for
  - stability of switched positive systems [Mason/Shorten ’06]
  - positivity preserving model reduction [Reis/V. ’08]

⇒ Need a suitable generalisation of Perron-Frobenius Theory to matrix pairs.
**Theorem (Classical Perron-Frobenius theorem)**

Let \( A \in \mathbb{R}^{n \times n} \). If \( A \geq 0 \),

then \( \rho(A) \) is an eigenvalue of \( A \) and there exists a corresponding nonnegative eigenvector.

**Theorem (Generalisation for matrix pairs [Mehrmann/Nabben/V. ’08])**

Let \((E, A)\) regular and set \( \hat{E} := (\lambda E - A)^{-1} E, \hat{A} := (\lambda E - A)^{-1} A \). If \( \hat{E}^D \hat{A} \geq 0 \),

then \( \rho_f(E, A) \) is an eigenvalue of \((E, A)\) and there exists a corresponding nonnegative eigenvector.
all finite eigenvalues of \((E, A)\) are eigenvalues of \(\hat{E}^D \hat{A}\)
all finite eigenvalues of \((E, A)\) are eigenvalues of \(\hat{E}^D \hat{A}\)
What is this matrix $\hat{E} D \hat{A}$

- all finite eigenvalues of $(E, A)$ are eigenvalues of $\hat{E} D \hat{A}$
- eigenvalue $\infty$ is “mapped” to 0
If $\hat{E}^D \hat{A} \geq 0$, then

- $\rho_f(E, A) = \rho(\hat{E}^D \hat{A})$ is an eigenvalue and
- there exists a corresponding nonnegative eigenvector.
Have: c-stable pair \((E, A)\)
Have: c-stable pair \((E, A)\)

But: \(\hat{E}^D \hat{A}\) is NOT c-stable since \(\infty\) is mapped to 0.
Have: c-stable pair \((E, A)\)

But: \(\hat{E}^D \hat{A}\) is NOT c-stable since \(\infty\) is mapped to 0.

Idea: Shift the eigenvalue 0 into \(\mathbb{C}^-\)
Lemma (Matrix shift [V. ’08])

If \((E, A)\) regular and c-stable, then for any \(\alpha > 0\) we have

\[
\tilde{M} := \hat{E}^D \hat{A} - \alpha (I - P_r)
\]

is a (regular) c-stable matrix.

Positive stable systems: all results carry over to DAEs [V. ’08]

<table>
<thead>
<tr>
<th>ODE</th>
<th>DAE</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\dot{x} = Ax + Bu)</td>
<td>(E \dot{x} = Ax + Bu)</td>
</tr>
</tbody>
</table>

\(-A\) is an \(M\)-matrix

\(-\tilde{M}\) is an \(M\)-matrix
Theorem (ODEs [Luenberger ’79])

\[ \dot{x} = Ax + Bu \text{ is positive iff } A \text{ is a } -Z\text{-matrix and } B \geq 0. \]

Theorem (DAEs [V. ’08])

Consider \( E \dot{x} = Ax + Bu \), \((E, A)\) regular of \( \text{ind}(E, A) = \nu \). Assume

(i) \( (I - P_r)(\hat{E} \hat{A}^D)^i \hat{A}^D \hat{B} \leq 0 \) for \( i = 0, \ldots, \nu - 1 \),

(ii) \( P_r \geq 0 \).

\( E \dot{x} = Ax + Bu \text{ is positive iff} \)

1. there exists a scalar \( \alpha \geq 0 \) such that the matrix

\[ \hat{M} := \hat{E}^D \hat{A} - \alpha(I - P_r) \]

is a \(-Z\)-matrix,

2. \( \hat{E}^D \hat{B} \geq 0 \),
Idea:
Approximate the system

\[ E \dot{x} = Ax + Bu, \; x(t_0) = x_0, \]
\[ y = Cx + Du. \]

with transfer function \( G(s) = C(sE - A)^{-1}B + D \) by a system

\[ \tilde{E} \dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{B}u, \; \tilde{x}(t_0) = \tilde{x}_0, \]
\[ \tilde{y} = \tilde{C}\tilde{x} + \tilde{D}u. \]

of order \( l \ll n \) with \( \|y - \tilde{y}\| = \|Gu - \tilde{G}u\| \leq \|G - \tilde{G}\||u| \) is small.

- especially useful for controller design
- important that system properties are preserved!
A stable system \((A, B, C, D)\) (with \(E = I\)) is called balanced, if solutions \(P, Q\) of the Lyapunov equations

\[
AP + PA^T + BB^T = 0, \quad A^TQ + QA + C^TC = 0,
\]
satisfy \(P = Q = \Sigma = \text{diag}(\sigma_1, \ldots, \sigma_k, 0 \ldots, 0)\), where

- \(k \leq n\) and
- \(\sigma_1 > \ldots > \sigma_k > 0\) are called \textit{Hankel singular values}.

\(\triangleright\) a balanced system can be obtained via state-space transformation

\(\triangleright\) states corresponding to small \(\sigma_i\) are difficult to reach and to observe \(\rightsquigarrow\) may be truncated
Balanced truncation

... truncation

\[ A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad C = \begin{bmatrix} C_1 & C_2 \end{bmatrix}, \quad D, \]
Balanced truncation

... truncation

\[ A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad C = \begin{bmatrix} C_1 & C_2 \end{bmatrix}, \quad D, \]

balanced truncation / singular perturbation balanced truncation

\[ \tilde{A} = A_{11}, \quad \tilde{A} = A_{11} - A_{12}A_{22}^{-1}A_{21}, \]
\[ \tilde{B} = B_1, \quad \tilde{B} = B_1 - A_{12}A_{22}^{-1}B_2, \]
\[ \tilde{C} = C_1, \quad \tilde{C} = C_1 - C_2A_{22}^{-1}A_{21}, \]
\[ \tilde{D} = D, \quad \tilde{D} = D - C_2A_{22}^{-1}B_2. \]
Balanced truncation

\[ A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad C = \begin{bmatrix} C_1 & C_2 \end{bmatrix}, \quad D, \]

balanced truncation / singular perturbation balanced truncation

\[ \tilde{A} = A_{11}, \quad \tilde{A} = A_{11} - A_{12}A_{22}^{-1}A_{21}, \]
\[ \tilde{B} = B_1, \quad \tilde{B} = B_1 - A_{12}A_{22}^{-1}B_2, \]
\[ \tilde{C} = C_1, \quad \tilde{C} = C_1 - C_2A_{22}^{-1}A_{21}, \]
\[ \tilde{D} = D, \quad \tilde{D} = D - C_2A_{22}^{-1}B_2. \]

\[ \square \]

the reduced order system is stable and balanced

\[ \| G - \tilde{G} \|_{H_\infty} \leq 2 \sum_{i=\ell+1}^{n} \sigma_i, \]

\[ (\| G \|_{H_\infty} := \sup_{\omega \in \mathbb{R}} \| G(i\omega) \|) \]
\[
\begin{align*}
\dot{x} &= Ax + Bu, \ x(0) = x_0, \\
y &= Cx + Du
\end{align*}
\]
\[
\Rightarrow \quad A = \begin{bmatrix} * & + \\ + & * \end{bmatrix}, \ B, \ C, \ D \geq 0
\]

**Problem**

State-space transformations to balanced form destroy positivity!
\[
\dot{x} = Ax + Bu, \quad x(0) = x_0, \quad \Rightarrow \quad A = \begin{bmatrix} \ast & \ast \\ \ast & \ast \end{bmatrix}, \quad B, \ C, \ D \geq 0
\]

\[
y = Cx + Du
\]

**Problem**

State-space transformations to balanced form destroy positivity!

**Allowed transformations**

monomial (in particular: positive diagonal)
\[ \dot{x} = Ax + Bu, \quad x(0) = x_0, \quad \Rightarrow \quad A = \begin{bmatrix} * & + \\ + & * \end{bmatrix}, \quad B, \quad C, \quad D \geq 0 \]

\[ y = Cx + Du \]

**Problem**

State-space transformations to balanced form destroy positivity!

**Allowed transformations**

monomial (in particular: positive diagonal)

**Solution** [Li/Paganini ’05]

One obtains the same error bound for Lyapunov inequalities

\[ AP + PA^T + BB^T \preceq 0, \quad A^T Q + QA + C^T C \preceq 0. \]
Diagonal solution of Lyapunov inequalities

Lemma (Reis/V. ’07)

For a stable \( A = \begin{bmatrix} * & + \\ + & * \end{bmatrix} \), i.e. \( A \) is \(-M\)-matrix, there exist \( P, Q \) positive diagonal, such that

\[
AP + PA^T + BB^T \preceq 0, \quad A^T Q + QA + C^T C \preceq 0,
\]

Proof. \(-M\)-matrices are diagonally stable, i.e. there exists \( X \) positive diagonal s.t.

\[
AX + XA^T \prec 0 \quad \Rightarrow \quad \exists \delta > 0 : \ AX + XA^T + \delta I \prec 0
\]

Let \( \epsilon := \|BB^T\| \), then \( \epsilon I - BB^T \) is symmetric \( M\)-matrix (Stieltjes) \( \Rightarrow \epsilon I - BB^T \) positive semidefinite, i.e. \( \epsilon I \succeq BB^T \). Then,

\[
A\tilde{X} + \tilde{X}A^T + BB^T :\succeq A\frac{\epsilon}{\delta}X + \frac{\epsilon}{\delta}XA^T + \epsilon I \prec 0.
\]
We have that

- there exists a (diagonal) state-space transformation to a positive balanced system

- balanced truncation (bt)/ singular perturbation balanced truncation (sp) preserve positivity:
  
  **(bt)** Submatrices of nonnegative/$(-Z)$-matrices are again nonnegative/$(-Z)$-matrices.

  **(sp)**

  \[
  \tilde{A} = A_{11} - A_{12} A_{22}^{-1} A_{21}, \quad \tilde{B} = B_1 - A_{12} A_{22}^{-1} B_2, \\
  \tilde{C} = C_1 - C_2 A_{22}^{-1} A_{21}, \quad \tilde{D} = D - C_2 A_{22}^{-1} B_2.
  \]

  $A_{22}$ is $-M$-matrix $\Rightarrow A_{22}^{-1} \leq 0 \Rightarrow$ Schur complements preserve nonnegativity/$(-Z)$-matrix properties!

- usual $H_\infty$ error bound available
System of $n$ water reservoirs (continuous-time); error bound 0.0167;

Leslie model of age-structured population (discrete-time); error bound 0.0357.

6.4. Examples

The frequency responses, i.e., the transfer function $G(s)$ at values $s = j\omega$, for $\omega \in [0, 3]$, of the original and of the reduced order models are depicted in the upper diagram of Figure 6.2. The lower diagram shows the frequency response of the error systems along with the mutual error bound 0.0162.

As an example in discrete-time, we consider the well-known Leslie model [79], which describes the time evolution of age-structured populations.
\[ E \dot{x} = Ax + Bu, \quad x(0) = x_0, \]
\[ y = Cx + Du \]

\[ \Rightarrow \text{Transfer function:} \]
\[ G(s) = C(sE - A)^{-1}B + D \]
\[ \begin{align*}
\dot{x} &= Ax + Bu, \quad x(0) = x_0, \\
y &= Cx + Du
\end{align*} \quad \Rightarrow \text{Transfer function:} \\
G(s) &= C(sE - A)^{-1}B + D
\]

**Problem**

\(G(s)\) may be improper and the \(H_\infty\)-norm is \(\infty\).
Model reduction for DAEs

\[ E \dot{x} = Ax + Bu, \quad x(0) = x_0, \quad y = Cx + Du \]

\[\Rightarrow\text{Transfer function:} \quad G(s) = C(sE - A)^{-1}B + D\]

**Problem**

\( G(s) \) may be improper and the \( H_\infty \)-norm is \( \infty \).

**Idea:** additively decompose

\[ G(s) = G_{sp}(s) + P(s) \]

- reduce strictly proper part \( G_{sp}(s) \) as in ODE case
- polynomial part \( P(s) \) remains unchanged

In the difference \( \| G - \tilde{G} \| \) the polynomial part cancels out

error bound as in the ODE case!
additive decomposition of transfer function and system matrices!

- reduction of the parts as in the ODE case
  - $G_{sp} \rightsquigarrow$ singular perturbation;
  - $P(s) \rightsquigarrow$ balanced truncation;

- show $P(s)$ remains unchanged

- definition of reduced-order spectral projector
  - $\tilde{P}_r := [P_r]_{11} - [P_r]_{12}[P_r]^{-1}_{22}[P_r]_{21}$;
  - show $\tilde{P}_r$ is nonnegative projector [Friedland/V.'08];

- reassembling to a reduced-order descriptor system, which
  - is positive
  - yields the usual error bound.
Positive systems

- state/output nonnegative for nonnegative input/initial value;

Results

- generalisation of the Perron-Frobenius Theory to matrix pairs \((E, A)\);
- characterisation of positive DAEs;
- matrix shift: all stability criteria carry over to the descriptor case;
- positivity preserving model reduction for ODEs and DAEs.
Thank you for your attention!!!