From (Deductive) Fuzzy Logic to (Logic-Based) Fuzzy Mathematics

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It is indisputable that mathematical structures arising around vague/ fuzzy/ non-bivalent concepts have a broad range of applications; therefore they have been intensively investigated during the last five decades. The discipline studying these structures is, maybe unfortunately, called *Fuzzy Mathematics*.

This discipline started by Zadeh's Fuzzy Set Theory [7] (although there are several almost forgotten predecessors) and already from its early days the role of logic have been noticed, stressed, and studied. However, fuzzy logic as a formal symbolic system (I will use the term 'deductive fuzzy logic' in this text) in the spirit of other non-classical logics has been thoroughly developed only recently. The paper [1] describes the deference between traditional and deductive fuzzy logic (refining Zadeh's original distinction between broad and narrow fuzzy logic). Very roughly speaking: deductive fuzzy logic deals with degrees of truth only, whereas traditional fuzzy logic speaks about also about degrees of belief, preference, entropy, necessity, etc. The consequences of this restriction narrow down the agenda of deductive fuzzy logic but give methodological clarity, determine the applicability scope and provide a research *focus* which leads to rapid development of the theory (and hopefully of some applications soon).

There is an ongoing project of the Prague research group in fuzzy logic, directed towards developing the *logic-based* fuzzy mathematics, i.e., an 'alternative' mathematics built in a formal analogy with classical mathematics, but using *deductive* fuzzy logic instead of the classical logic. The core of the project is a formulation of certain formalistic methodology (see [4]), a proposed foundational theory (see [3, 5]), and development of the particular disciplines of fuzzy mathematics within the foundational theory using the formalistic methodology. The proposed foundational theory is called Fuzzy Class Theory (FCT) and it is a first-order theory over multi-sorted predicate fuzzy logic, with a very natural axiomatic system which approximates nicely Zadeh's original notion of fuzzy set. In paper [4] the authors claim that the whole enterprize of Fuzzy Mathematics can be formalized in FCT. This is still true as classical logic is formally interpretable inside deductive fuzzy logic, however in the parts of fuzzy mathematics 'incompatible' with the requirements of deductive fuzzy logic our approach provides (very) little added value.

An important feature of the theory is the gradedness of all defined concepts, which makes it more genuinely fuzzy than traditional approaches. Indeed, e.g. in the theory of fuzzy relations the majority of traditional characterizing properties, such as reflexivity, symmetry, transitivity, and so forth, are defined in a strictly crisp way, i.e., as properties that either hold fully or do not hold at all (the notion of fuzzy inclusion is a notable exception; graded properties of fuzzy relations were originally studied by Siegfried Gottwald in [6]). One may be tempted to argue that it is somewhat peculiar to fuzzify relations by allowing intermediate degrees of relationships, but, at the same time, to still enforce strictly crisp properties on fuzzy relations. This particularly implies that all results are effective only if some assumptions are fully satisfied, but say nothing at all if the assumptions are only fulfilled to a certain degree (even if they are *almost* fulfilled).

In this talk I start by formulating and explaining the restrictions of the deductive fuzzy logic and presenting advantages (and disadvantages) of such restrictions. Then I sketch the methodology and formalism of FCT and illustrate it using simple examples from the theory of fuzzy relation (from the paper [2]). Finally I put this approach in the context of other nonclassical-logic-based mathematics (intuitionistic, relevant, substructural, etc.); compare logic-based, categorial, and traditional fuzzy mathematics; and address the possible outlooks of FCT and its role in future fuzzy mathematics.

References

1. Libor Běhounek. On the difference between traditional and deductive fuzzy logic. *Fuzzy Sets and Systems*, 159(10):1153–1164, 2008.
2. Libor Běhounek, Ulrich Bodenhofer, and Petr Cintula. Relations in Fuzzy Class Theory: Initial steps. *Fuzzy Sets and Systems*, 159(14):1729–1772, 2008.
3. Libor Běhounek and Petr Cintula. Fuzzy class theory. *Fuzzy Sets and Systems*, 154(1):34–55, 2005.
4. Libor Běhounek and Petr Cintula. From fuzzy logic to fuzzy mathematics: A methodological manifesto. *Fuzzy Sets and Systems*, 157(5):642–646, 2006.
5. Libor Běhounek and Petr Cintula. Fuzzy Class Theory: A primer v1.0. Technical Report V-939, Institute of Computer Science, Academy of Sciences of the Czech Republic, Prague, 2006. Available at www.cs.cas.cz/research/library/reports_900.shtml.
6. Siegfried Gottwald. *Fuzzy Sets and Fuzzy Logic: Foundations of Application—from a Mathematical Point of View.* Vieweg, Wiesbaden, 1993.
7. Lotfi A. Zadeh. Fuzzy sets. *Information and Control*, 8(3):338–353, 1965.