Conditionals and independence in many-valued logics

DANIELE MUNDICI

Department of Mathematics, University of Florence
Viale Morgagni 67/a, 50134 Florence, Italy
mundici@math.unifi.it

Formulas and valuations in boolean logic are a traditional source of examples of "e-vents" and "possible worlds". However, many events of interest in everyday life are more general than yes-no events, as described in boolean logic. Their possible outcomes typically range over a continuous spectrum, which after a suitable normalization can be restricted within the unit real interval $[0,1]$.

Events and possible worlds from physical systems States and observables of physical systems provide a very general source of continuously valued events and possible worlds. Let SYST be a "physical system". Following [6, pp.362–369], a rigorous account of SYST is given by its associated C^*-algebra A, with the set $\mathsf{A}_{\mathrm{sa}} \subseteq \mathsf{A}$ of self-adjoint elements, and the set $\mathsf{S}^* \subseteq \mathbb{R}^{\mathsf{A}_{\mathrm{sa}}}$ of real-valued normalized positive linear functionals on A_{sa}.[1] For any $W \in \mathsf{S}^*$ and $X \in \mathsf{A}_{\mathrm{sa}}$ the real number $W(X)$ is *the expectation value of the observable X when* SYST *is prepared in mode W.*

Given a set $E = \{X_1,\ldots,X_m\}$ of nonzero positive elements of A_{sa}, W determines, by normalization, the map $w\colon E \to [0,1]$ given by $w(X_i) = w(X_i)/||X_i||$, where $||X_i||$ is the norm of X_i. Intuitively, the *event* X_i says "the observable X_i has a high value," and w evaluates how true X_i is. The set $\mathsf{W} \subseteq [0,1]^E$ of *possible worlds* is defined by $\mathsf{W} = \{w \mid W \in \mathsf{S}^*\}$. W is a closed nonempty set in the cube $[0,1]^E = [0,1]^n$.

Coherent bets on E and W Having thus presented a sufficiently general framework for the notions of "event" and "possible world", we will now consider two *abstract* sets $E = \{X_1,\ldots,X_m\}$ and $\mathsf{W} \subseteq [0,1]^{\{X_1,\ldots,X_m\}} = [0,1]^n$, without any reference to observables and states of physical systems.

Suppose two players Ada (the bookmaker) and Blaise (the bettor) wager money on the outcome of events $X_1,\ldots,X_m$ within a prescribed set W of possible worlds. By definition, Ada's book is a map $\beta\colon E \to [0,1]$, containing a "betting odd" $\beta(X_i)$ for each event. Blaise, who knows β, chooses a "stake" $\sigma_i \in \mathbb{R}$ for each $i = 1,\ldots,m$: by definition, σ_i is the amount of money (measured in euro for definiteness) to be paid to the bettor if event X_i occurs. Money transfers are oriented in such a way that "positive" means Blaise-to-Ada. For each $i = 1,\ldots,m,\ \sigma_i \cdot \beta(X_i)$ euro are paid, with the proviso that $-\sigma_i \cdot v(X_i)$ euro will be paid back, in the possible world $v \in \mathsf{W}$. Any stake $\sigma_i < 0$ results in a sort of "reverse bet", where the bookmaker-bettor roles are interchanged: Ada first pays Blaise $|\sigma_i| \cdot \beta(X_i)$ euro, and Blaise will pay back $|\sigma_i| \cdot v(X_i)$ in the possible world v.

[1] SYST is said to be classical if its associated C^*-algebra is commutative.

Ada's book β would lead her to financial disaster if Blaise could choose stakes $\sigma_1,\ldots,\sigma_m$ ensuring him to win at least one million euro in every possible world. Replacing the word "disaster" by "incoherence", we have the following definition:

> A map $\beta\colon E \to [0,1]$ is W-*incoherent* if for some $\sigma_1,\ldots,\sigma_m \in \mathbb{R}$ we have $\sum_{i=1}^{m} \sigma_i \cdot (\beta(X_i) - v(X_i)) < 0$ for all $v \in \mathsf{W}$. Otherwise, β is W-*coherent*.

In the particular case when $\mathsf{W} \subseteq \{0,1\}^E$, we obtain De Finetti's no-Dutch-Book criterion for coherent probability assessments of yes-no events (see [3, §7, p. 308], [4, pp. 6-7], [5, p. 87]).

The role of Łukasiewicz logic and MV-algebras in $[0,1]$-valued probability We refer to [2] for background on Łukasiewicz (always propositional and infinite-valued) logic Ł$_\infty$, and MV-algebras. We will denote by $\odot, \oplus, \neg$ the connectives of conjunction, disjunction and negation. F_m is the set of all formulas whose variables are in the set $\{X_1,\ldots,X_m\}$. A *(Łukasiewicz) valuation of* F_m is a function $v\colon \mathsf{F}_m \to [0,1]$ such that

$$v(\neg\phi) = 1 - v(\phi),\ v(\phi \oplus \psi) = \min(1, v(\phi) + v(\psi)),\ v(\phi \odot \psi) = \max(0, v(\phi) + v(\psi) - 1)$$

for all $\phi, \psi \in \mathsf{F}_m$. A formula is a *tautology* if it is satisfied by every valuation[2]. We say that *v satisfies a set of formulas* $\Psi \subseteq \mathsf{F}_m$ if $v(\theta) = 1$ for all $\theta \in \Psi$. Ψ is *consistent* if it is satisfied by at least one valuation. A formula θ is *consistent* if so is $\{\theta\}$. Two formulas $\phi, \psi \in \mathsf{F}_m$ are Ψ-*equivalent* if from Ψ one obtains $\phi \leftrightarrow \psi$ (i.e., $(\neg\phi \oplus \psi) \odot (\neg\psi \oplus \phi)$) using all tautologies and modus ponens. We denote by $\frac{\psi}{\equiv_\Psi}$ the Ψ-equivalence class of formula ψ.

As is well known, MV-algebras stand to Ł$_\infty$ as boolean algebras stand to classical two-valued propositional logic. Thus for instance, the set of Ψ-equivalence classes of formulas forms the MV-algebra

$$\mathsf{L}(\Psi) = \frac{\mathsf{F}_m}{\equiv_\Psi} = \left\{ \frac{\phi}{\equiv_\Psi} \mid \phi \in \mathsf{F}_m \right\}.$$

Part of the proof of the following theorem is in [11]. The rest will appear elsewhere.

Theorem 1. *For any set $E = \{X_1,\ldots,X_m\}$ and closed nonempty set $W \subseteq [0,1]^E = [0,1]^{\{1,\ldots,m\}} = [0,1]^m$, there is a set Θ of formulas in the variables $X_1,\ldots,X_m$ such that W coincides with the set of restrictions to E of all valuations satisfying Θ. Further, a map $\beta\colon E \to [0,1]$ is W-coherent iff it can be extended to a convex combination of valuations satisfying Θ iff there is a* state *s of $\mathsf{L}(\Theta)$ such that $\beta(X_i) = s(X_i/\equiv_\Theta)$, for all $i = 1,\ldots,m$.*

As the reader will recall, a *state* of an MV-algebra B is a map $s\colon B \to [0,1]$ such that $s(1) = 1$ and $s(x \oplus y) = s(x) + s(y)$ whenever $x \odot y = 0$. We say that s is *faithful* if $s(x) = 0$ implies $x = 0$. We say that s is *invariant* if $s(\alpha(x)) = s(x)$ for every automorphism α of B and element $x \in B$.

[2] it is tacitly understood that all valuations and formulas are of F_m.

Under the restrictive hypothesis $\mathsf{W} \subseteq \{0,1\}^E$ the above theorem boils down to De Finetti's well known characterization of coherent assessments of yes-no events (see [3, pp.311-312], [4, Chapter 1], [5, pp.85-90]).

De Finetti's theorem was extended by Paris [15] to several modal logics, by Kühr et al., [9] to all $[0,1]$-valued logics whose connectives are *continuous*, including all finite-valued logics. In their paper [1], Aguzzoli, Gerla and Marra further extend De Finetti's criterion to Gödel logic [7], a logic with a *discontinuous* implication connective. By Theorem 1, the various kinds of "events", "possible worlds" and "coherent probability assessments" arising in all these logic contexts can be re-interpreted in Łukasiewicz logic.

Conditionals and their invariance Given the universal role of Łukasiewicz logic and MV-algebraic states for the treatment of coherent probability assessments, one is naturally led to develop a theory of conditionals in this logic. In [12, 3.1-3.2], the present author gave the following definition:

> A *conditional* is a map $\mathsf{P}\colon \theta \mapsto \mathsf{P}_\theta$ such that, for every $m = 1,2,\ldots$ and every consistent formula $\theta \in \mathsf{F}_m$, P_θ is a state of the MV-algebra $\mathsf{L}(\{\theta\})$. We say that P is *invariant* if for any two consistent formulas $\phi \in \mathsf{F}_m$, $\psi \in \mathsf{F}_n$, and isomorphism η of $\mathsf{L}(\{\phi\})$ onto $\mathsf{L}(\{\psi\})$, we have $\mathsf{P}_\phi = \mathsf{P}_\psi \circ \eta$, where $\circ$ denotes composition. P is said to be *faithful* if so is every state P_θ.

The main result of [12] is

Theorem 2. *Łukasiewicz logic $Ł_\infty$ has a faithful invariant conditional* P^*.

It follows that P^* is invariant under equivalent reformulations of the same event. In more detail:

Corollary (a). *For every formula $\psi \in \mathsf{F}_m$ let us write $\mathsf{P}^*{}_\theta(\psi)$ instead of $\mathsf{P}^*{}_\theta(\frac{\psi}{\equiv_{\{\theta\}}})$, and say that $\mathsf{P}^*{}_\theta(\psi)$ is* the probability of ψ given θ. *We then have*

$$\mathsf{P}^*_{\psi\leftrightarrow\psi}(\psi) = \mathsf{P}^*_{\psi\leftrightarrow X_{m+1}}(X_{m+1}). \tag{1}$$

Proof. We assume familiarity with [12] and [2]. A *rational polyhedron* in $[0,1]^n$ is a finite union of simplexes in $[0,1]^n$, such that the coordinates of the vertices of each simplex are rational.

Given rational polyhedra $P \subseteq \mathbb{R}^n$ and $Q \subseteq \mathbb{R}^m$ by a $\mathbb{Z}$*-homeomorphism* we understand a piecewise linear homeomorphism η of P onto Q such that each linear piece of both η and η^{-1} has integer coefficients.

We denote by f_ψ the McNaughton function of ψ, (see e.g., [2, p.221]). Let the rational polyhedron $D \subseteq [0,1]^{m+1}$ be defined by

$$D = \{(x_1,\ldots,x_{m+1}) \in [0,1]^{m+1} \mid x_{m+1} = f_\psi(x_1,\ldots,x_m)\}.$$

Up to isomorphism, the MV-algebra $\mathsf{L}(\{\psi \leftrightarrow \psi\})$ of the tautology $\psi \leftrightarrow \psi$ is the free m-generator MV-algebra $Free_m$, i.e., (by McNaughton theorem [2, 9.1.5]) the MV-algebra $\mathsf{M}([0,1]^m)$ of all McNaughton functions $f\colon [0,1]^m \to [0,1]$.

By a routine variant of [12, 2.3], the MV-algebra $\mathsf{L}(\{\psi \leftrightarrow X_{m+1}\})$ is the MV-algebra

$$\mathsf{M}(D) = \{l \upharpoonright D \mid l \in \mathsf{M}([0,1]^{m+1})\}$$

obtained by restricting to D the McNaughton functions of $Free_{m+1} = \mathsf{M}([0,1]^{m+1})$.

We can safely use the identifications

$$\frac{\psi}{\equiv_{\{\psi \leftrightarrow \psi\}}} = f_\psi \qquad \text{and} \qquad \frac{X_{m+1}}{\equiv_{\{X_{m+1} \leftrightarrow \psi\}}} = \pi_{m+1} \upharpoonright D, \tag{2}$$

where $\pi_{m+1} \colon [0,1]^{m+1} \to [0,1]$ is the $(m+1)$th coordinate function

$$\pi_{m+1}(x_1, \ldots, x_{m+1}) = x_{m+1}.$$

In view of Theorem 2, to conclude the proof, we must only construct an isomorphism of $\mathsf{M}([0,1]^m)$ onto $\mathsf{M}(D)$ sending f_ψ to the coordinate function π_{m+1}. The map

$$\eta \colon (x_1, \ldots, x_m) \mapsto (x_1, \ldots, x_m, f_\psi(x_1, \ldots, x_m))$$

is promptly seen to be a $\mathbb{Z}$-homeomorphism of $[0,1]^m$ onto D. The inverse map projects D onto the face of $[0,1]^{m+1}$ given by $x_{m+1} = 0$. In symbols,

$$\eta^{-1} = (\pi_1, \ldots, \pi_m) \upharpoonright D.$$

The map

$$\Omega \colon g \in \mathsf{M}(D) \mapsto g \circ \eta \in \mathsf{M}([0,1]^m)$$

is a one-one homomorphism of $\mathsf{M}(D)$ into $\mathsf{M}([0,1]^m)$. The map

$$\mho \colon h \in \mathsf{M}([0,1]^m) \mapsto h \circ \eta^{-1} \in \mathsf{M}(D)$$

is a one-one homomorphism of $\mathsf{M}([0,1]^m)$ into $\mathsf{M}(D)$. Trivially, these two maps are inverse of each other, and in view of (2) we can write

$$\mho \colon \mathsf{L}(\{\psi \leftrightarrow \psi\}) = \mathsf{M}([0,1]^m) \cong \mathsf{M}(D) = \mathsf{L}(\{\psi \leftrightarrow X_{m+1}\}).$$

The two elements $\frac{\psi}{\equiv_{\{\psi \leftrightarrow \psi\}}}$ and $\frac{X_{m+1}}{\equiv_{\{X_{m+1} \leftrightarrow \psi\}}}$ correspond under the isomorphism $\mho$. This completes the proof. □

A supplementary analysis of the proof of the main theorem of [12] shows that

$$\mathsf{P}^*_{\psi \leftrightarrow X_{m+1}}(X_{m+1}) = \mathsf{P}^*_{\psi \leftrightarrow \psi}(\psi) = \int_{[0,1]^m} f_\psi. \tag{3}$$

More generally, a similar argument proves:

Corollary (b). *For any formula* ψ *and consistent formula* θ *we have*

$$\mathsf{P}^*_\theta(\psi) = \mathsf{P}^*_{\theta \odot (\psi \leftrightarrow X)}(X), \tag{4}$$

provided the variable X does not occur in θ and ψ.

We are now in a position to introduce a reasonable notion of independence, by saying that a formula α is P^*-*independent of* (a consistent formula) θ if the probability of α given θ coincides with the unconditional probability of α. In view of Corollary (a) we can equivalently write

$$\mathsf{P}^*_\theta(\alpha) = \mathsf{P}^*_{\theta\leftrightarrow\theta}(\alpha) = \mathsf{P}^*_{\alpha\leftrightarrow\alpha}(\alpha) = \mathsf{P}^*_{X\leftrightarrow\alpha}(X), \tag{5}$$

where X is a fresh variable. As a consequence of Corollary (b) we have

Corollary (c). *If α and θ are two formulas in disjoint sets of variables $\{Y_1,\dots,Y_m\}$ and $\{Z_1,\dots,Z_n\}$, and θ is consistent, then α is P^*-independent of θ.*

Concluding remarks For any finite set $E = \{X_1,\dots,X_n\}$ whose elements are called "events", and closed set $W \subseteq [0,1]^E$ whose elements are called "possible worlds", following De Finetti we have defined a bookmaker's map $b\colon E \to [0,1]$ to be W-incoherent if a bettor can fix (positive or negative) stakes $s_1,\dots,s_n$ ensuring him a profit of least one million euro (equivalently, a profit > 0) in any possible world of W.

No matter the physical or logical nature of E and W, Theorem 1 shows that there is a theory Θ in Łukasiewicz logic such that W-coherent maps coincide with restrictions to E of states of the MV-algebra $\mathsf{L}(\Theta)$. In particular, when W is a set of valuations in *any* $[0,1]$-valued logic L, and E is a set of formulas in L, W-coherent maps on E can always be interpreted in Łukasiewicz logic.

It is often claimed that De Finetti's coherence criterion yields an axiomatic approach to finitely additive probability measures, missing the full strength of Kolomgorov axioms. We beg to dissent: by the Kroupa-Panti theorem [8, 13], in every MV-algebra A—whence in particular in every boolean algebra— the set of (*finitely additive*) states of A is in canonical one-one correspondence with the set of (*countably additive*) regular Borel probability measures on the maximal spectrum of A. Thus the theory of finitely additive measures (i.e., states) on boolean algebras has the same degree of generality as the theory of regular Borel measures on their Stone spaces. Passing to the much larger class of MV-algebras, Theorem 1 in combination with the Kroupa-Panti theorem shows that De Finetti's coherence criterion has the same degree of generality as the theory of regular probability Borel measures on *any compact space*.

Theorem 2 shows that Łukasiewicz logic has a faithful invariant conditional P^*. In Corollary (a)-(b) a new result is proved, to the effect that P^* *does not make any distinction between (i) the probability of ψ given θ, and (ii) the probability of (the event described by) a fresh variable X given θ together with the information that X is equivalent to ψ.* A novel notion of independence is built on P^*, having various desirable properties, some of which are summarized in (5) and in Corollary (c).

For further information on MV-algebraic probability theory, including other approaches to conditional probability and independence, see [14, Chapters 20-22].

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